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A NEW LINKAGE FOR DESCRIBING A STRAIGHT LINE BY CONTINUOUS MOTION.

By JOHN J. QUINN, Ph. D., Pittsburg College, Pittsburg, Pa.

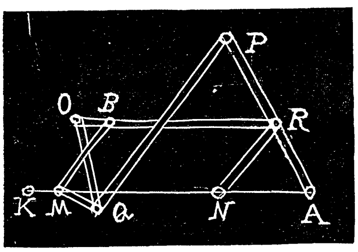
In the linkage given herewith the following conditions obtain: M and N are fixed points to which the whole system is pivoted.

$$MQ = OB = \tfrac{1}{2}OQ;$$

$$OQ = MB = PR = RA = NR = \tfrac{1}{2}QR;$$

$$PQ = OR;\ BR = MN.$$

It is required to find the locus of P and A for every movement of the system if all movable points except P and A describe circles.



1. From the stated conditions, MB is parallel to NR. Hence $\angle OBM = \angle BRN = \angle RNA$. Also $\triangle MOB$ is similar to $\triangle OPR$, since the respective sides are proportional.

Then $\angle OBM = \angle ORP$, being homologous angles of similar triangles. Hence $\angle PRN = 2\angle RNA$. Hence it follows that the locus of P is a straight line perpendicular to MN.

2. Since P moves in a straight line, and R is the mid-point of PA, it follows that the locus of A is a straight line. The following corollaries follow immediately:

1. The locus of any point on PA, except R, the mid-point, is an ellipse.
2. On a straight line through MN lay off $MK = MQ$, then K is a fixed point equidistant from the point O.
3. If the link OQ be unfastened at Q, and the point Q fixed at K, the loci of P and A remain unchanged.

THE PERFECT MAGIC SQUARES FOR 1908.*

By DR. G. B. M. ZERR, Philadelphia, Pa.

By Perfect Magic Squares are meant those made up of successive integral numbers. Suppose we denote the number of integers on a side of the Magic Square by n; then, if we start with unity, and write the numbers successively until the square is filled, in magic arrangement, each column, each row, and each of the two diagonals of the square will add to a sum equal to $\frac{n^3+n}{2}$. Whenever $\frac{1908-(n^3+n)/2}{n}$ is an integer we can form a perfect magic square for 1908.

$\frac{1908-(n^3+n)/2}{n}+1=\frac{3816-n^3+n}{2n}$ = the least integer permissible for forming the square. $\frac{1908-(n^3+n)/2}{n}+n^2=\frac{3816-n+n^3}{2n}$ = the greatest integer permissible for forming the square.

The only values of n for 1908 are 3, 8, and 9.

For $n=3$, least integer $=632$, greatest integer $=640$.

For $n=8$, least integer $=207$, greatest integer $=270$.

For $n=9$, least integer $=172$, greatest integer $=252$.

The three magic squares written out are as follows; all of which sum magically 1908.

$n=3$.

639	632	637
634	636	638
635	640	633

$n=8$.

207	269	209	267	266	212	264	214
262	216	260	218	219	257	221	255
223	253	225	251	250	228	248	230
246	232	244	234	235	241	237	239
238	240	236	242	243	233	245	231
247	229	249	227	226	252	224	254
222	256	220	258	259	217	261	215
263	213	265	211	210	268	208	270

*We pay our respects to the year just gone, through Dr. Zerr's article, and trust that some of our readers may do as much for 1909. EDITORS.

$n=9.$

218	229	240	251	172	183	194	205	216
228	239	250	180	182	193	204	215	217
238	249	179	181	192	203	214	225	227
248	178	189	191	202	213	224	226	237
177	188	190	201	212	223	234	236	247
187	198	200	211	222	238	235	246	176
197	199	210	221	232	243	245	175	186
207	209	220	231	242	244	174	185	196
208	219	230	241	252	173	184	195	206

ON THE EQUATION $x^n+y^n=nz^n$.*

By DR. JACOB WESTLUND, Purdue University.

E. Maillet† has proved that the equation

$$x^n+y^n=nz^n$$

is not solvable in rational integers, when n is a *regular*‡ prime. The object of the following note is to prove the impossibility of the above equation for any odd prime n, when z is not divisible by n.

If z is not divisible by n, it follows that neither x nor y can be divisible by n. We may also, without loss of generality, assume x and y relatively prime. Now it can easily be shown that x^n+y^n is either not divisible by n or is divisible by n^2 at least. For we have

*Read before the Chicago Section of the American Mathematical Society, March, 1908.

†*Annali di Matematica,* Series III, Vol. 12. See also, *Acta Matematica,* Vol. 24.

‡A prime n is said to be a *regular* prime, when it is not a factor of the numerator of any one of the first $(n-3)/2$ Bernoullian numbers.

$$x^n+y^n=(x+y-y)^n+y^n$$
$$=(x+y)^n-n(x+y)^{n-1}y+\dots+n(x+y)y^{n-1},$$

and since n is an odd prime, it follows that, if x^n+y^n is divisible by n, $x+y$ is also divisible by n, and hence x^n+y^n is divisible by n^2. From this follows directly the impossibility of the equation $x^n+y^n=nz^n$ when z is prime to n.

AN INTERESTING CLASS OF MONOTONIC FUNCTIONS.*

By ARTHUR R. SCHWEITZER, Chicago, Illinois.

1. In the *Rivista di Matematica*, Vol. 2, p. 43, Professor Peano has defined the following monotonic function. Representing the variable x of the interval (0, 1) by the series†

$$x=\sum_{k=1}^{+\infty}\frac{a_k}{m^k} \qquad m \overline{\gt} 2,\ a_k \overline{\gt} 1,$$

then he defines

$$t(x)=\sum_{k=1}^{+\infty}\left(\frac{a_k}{m^k}\right)^{nk}$$

where the n_k are integers $\overline{\gt} 2$. In this paper the function of Peano is generalized in such a manner that a very large class of monotonic functions is obtained which includes other types than the preceding; also an explicit evaluation of certain properties is given.

2. As is well known,‡ every point $0 \overline{\lt} x \overline{\lt} 1$ can be represented by the series

$$x=\sum_{k=1}^{+\infty}\frac{a_k}{\Pi_k}$$

where

$$\Pi_k=m_1\ m_2 \dots m_k \qquad m_k \overline{\gt} 2.$$

The point x has a unique representation when $x=0$ or $x=1$ or $x=r/s$, where r and s are relatively prime and s is not a divisor of the product Π_k for any k. In all other points x is irrational, and therefore has a unique represen-

*In this paper references will be made by notes to the following works: (1) Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, (2) Jordan's *Cours d' Analyse*, Vol. I, (3) Chrystal's *Algebra*. These will be briefly referred to as "Hobson," "Jordan," and "Chrystal," respectively.

†Chrystal I, p. 165. Representing X by the radix fraction $a, a_2, a_3 \dots a_n \dots$ then we obtain $t(x)$ by the suitable interpolation of zeros, *e. g.*, $0\,a_1,\ 0\,a_2,\ 0\,a_3 \dots 0\,a_n \dots$

‡Chrystal, pp. 165, 166.

tation, or x is expressible by a finite number of terms and hence has a double representation. In the latter case x has the repetend zero $(\dot{0})$ or the repetend (m_k-1) for an assigned k $(m_k'-1)$. Accordingly, we denote the points of the interval (0, 1) by $x=x_f$ and $x=x_\phi$, respectively. If $x=x_f$ and $x=x_\phi$ we define $x=x_\theta$. Thus we have the three sets of representations,

$$X_f=[x_f],\quad X_\phi=[x_\phi],\quad X_\theta=[x_\theta].$$

Hence, in the well known notation of Cantor,*

$$X_\theta=M(X_f,\ X_\phi),$$

and $\beta(X_f,\ X_\phi)$ is the set of representations of points admitting expression by an infinite radix fraction only. We denote further

$$R(X_f,\ X_\phi)=M(X_f,\ X_\phi)-\beta(X_f,\ X_\phi),$$

so that $R(X_f,\ X_\phi)$ is the set of representations of points expressible as a finite radix fraction.

3. We proceed to define the functions $f(x)$, $\phi(x)$, $\theta(x)$. If $x=x_f$ we define

$$f(x)=\sum_{k=1}^{+\infty} rx\frac{a_k^{l_k}}{\Pi'_k}$$

where

$$\Pi'_k=m'_1,\ m'_2,\ \ldots,\ m'_k,\qquad m'_k\geqq 2,$$

the numbers l_k are positive integers $\geqq 1$, and

$$m_k\leqq m'_k,\quad (m_k-1)^{l_k}\leqq m'_k-1;$$

also the numbers r_k are positive. The function $f(x)$ is then monotonic† if the numbers r_k form a non-increasing series. In an analogous manner we

$$f(x)=\sum_{k=1}^{+\infty}\frac{a_k}{(m_1+1)(m_2+1)\ldots(m_k+1)}.$$

define $f(x)$ and $\theta(x)$ corresponding to $x=x_\phi$ and $x=x_\theta$. Thus $f(x)$ and $\phi(x)$ are single-valued, but $\theta(x)$ is not. Denoting

$$F=[f(x)],\quad \Phi=[\phi(x)],\quad \Theta=[\theta(x)],$$

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*Hobson, §114; G. Cantor, *Matematische Annalen*, 17, p. 355.

†Hobson, p. 245, §189. In verifying the argument of the text the reader is requested to select from one of the numerous functions defined above a particular example, *e. g.*,

we have

$$\Theta = M(F, \Phi),$$

and corresponding to $\beta(X_f, X_\phi)$ and $R(X_f, X_\phi)$ we have $\beta(F, \Phi)$, $R(F, \Phi)$. As particular cases of the functions $f(x)$, $\phi(x)$, $\theta(x)$ defined above, in addition to the preceding function of Peano, we may mention the functions of G. Cantor, *Matematische Annalen*, 21, p. 590, and Liouville, *Journal de Matematica*, 16, p. 133.

4. In this, and the following sections, we derive explicitly some of the properties of $f(x)$, $\phi(x)$, $\theta(x)$ which we know in advance exist.* The numbers $f(x \pm 0)$, $\phi(x \pm 0)$ are easily evaluated by a suitable selection of sequences. We have, in any point $0 < x_0 < 1$.

$$f(x_0 - 0) = \phi(x_0 - 0),$$
$$f(x_0 + 0) = \phi(x_0 + 0).$$

If x_0 is a point of $R(X_f, X_\phi)$ then

$$f(x_0 - 0) = \phi(x_0),$$
$$f(x_0 + 0) = f(x_0).$$

If x_0 is a point of $\vartheta(X_f, X_\phi)$ then

$$f(x_0 - 0) = f(x_0),$$
$$f(x_0 + 0) = f(x_0).$$

Thus the functions $f(x)$ and $\phi(x)$ are discontinuous in every point expressible as a finite radix fraction and continuous in every point expressible as an infinite radix fraction only.

5. The set Θ of functional values $\theta(x)$ is nowhere dense and perfect.† Hence the Cantor content‡ of Θ is the difference between the length of the bounding interval and the sum of the point free intervals. For x on the set $R(X_f, X_\phi)$ we have

$$f(x) - \phi(x) = r_\mu \cdot \frac{a_\mu l_\mu - (a_\mu - 1) l_\mu}{\Pi'_\mu} - \sum_{k=1}^{+\infty} r_{\mu+k} \frac{(m_k - 1) l_{\mu+k}}{\Pi'_{\mu+k}},$$

where

$$x = \sum_{k=1}^{\mu} \frac{a_k}{\Pi_k} \qquad a_\mu \neq 0.$$

*Jordan, §71.
†Hobson, §72.
‡Hobson, §85.

Since the number of these intervals is $\Pi_{\mu-1}$, where $\Pi_0=1$, we easily find the sum of the point free intervals to be

$$\sum_{\mu=1}^{+\infty}\left[\frac{\Pi_{\mu-1}}{\Pi'_{\mu-1}}\cdot\sum_{k=0}^{+\infty} r_{\mu+k}\cdot\frac{(m_{\mu+k}-1)\,l_{\mu+k}}{m'_{\mu}\ldots m'_{\mu+k}}-\frac{\Pi_\mu}{\Pi'_\mu}\cdot\sum_{k=1}^{+\infty} r_{\mu+k}\frac{(m_{\mu+k}-1)\,l_{\mu+k}}{m'_{\mu+1}\ldots m'_{\mu+k}}\right]$$

$$=\sum_{k=1}^{+\infty} r_k\cdot\frac{(m_k-1)\,l_k}{\Pi'_k}-\operatorname*{Limit}_{\mu=+\infty}\frac{\Pi_\mu}{\Pi'_\mu}\sum_{k=1}^{+\infty} r_{\mu+k}\cdot\frac{(m_{\mu+k}-1)\,l_{\mu+k}}{m'_{\mu+1}\ldots m'_{\mu+k}}.$$

Hence the content of Θ is

$$C=\operatorname*{Limit}_{\mu=+\infty}\frac{\Pi_\mu}{\Pi'_\mu}\cdot\sum_{k=1}^{+\infty} r_{\mu+k}\cdot\frac{(m_{\mu+k}-1)\,l_{\mu+k}}{m'_{\mu+1}\ldots m'_{\mu+k}}.$$

Thus, if $m'_k=m_k^{n_k}$, $m_k \overline{\gtrless} 2$, the corresponding sets Θ have content zero; a similar result is obtained in the case of the Liouville and Cantor functions mentioned above. Also Θ has content zero if the series $r_1+r_2+r_3+$ etc., is convergent. Again, if

$$\begin{aligned}
&r_{\mu+k}=1,\\
&l_{\mu+k}=1,\\
&m_{\mu+k}=(\mu+k+1)^2-1,\\
&m'_{\mu+k}=(\mu+k+1)^2,
\end{aligned}$$

the content of Θ is the infinite product

$$\tfrac{3}{4}\cdot\tfrac{8}{9}\cdot\tfrac{15}{16}\ldots$$

which has a value different from zero.

6. When Θ has content zero we can immediately rectify the curve of the function $f(x)$ in the sense defined by Scheeffer,* *Acta Mathematica*, Vol. 5, p. 59. We require the total discontinuity function of $f(x)$, say $\Omega(x_0, x)$. This function has for $x_0 \overline{\gtrless} x \overline{\gtrless} 1$ the value given by the sum of the point free intervals of the subset of Θ contained in the closed interval $[f(x_0), \phi(x)]$, plus the discontinuities of $f(x)$ and $\phi(x)$ in the points x_0, x. We have

*Jordan, p. 90 and following.

$$\Omega(x_0, x)=[\phi(x)-f(x_0)]+[f(x_0)-\phi(x_0)]+[f(x)-\phi(x)]=f(x)-\phi(x_0).$$

Hence $$f(x)-\Omega(x_0, x)=\phi(x_0).$$

Corresponding to the closed interval (x_0, x_1) the length of the function $f(x)-\Omega(x_0, x)$ is x_1-x_0 and hence the length of the curve of $f(x)$ for this interval is

$$L=x_1-x_0+f(x_1)-\phi(x_0).$$

7. Finally, we determine the integral*

$$\int_0^1 f(x)\,dx.$$

We divide the interval (0, 1) into Π_v equal parts. Then every partition point assumes the form

$$x_{\cdot e_1 e_2 \dots e_v}=\sum_{k=1}^{v}\frac{e_k}{\Pi_k},$$

and conversely, every radix fraction so expressible is a partition point. Denoting the length of the interval whose first point is $x_{\cdot e_1 e_2 \dots e_v}$ by $\delta_{\cdot e_1 e_2 \dots e_v}$ we obtain,

$$\int_0^1 f(x)\,dx=\underset{v=+\infty}{\text{Limit}}\sum_{(e_k)} f(x_{\cdot e_1 e_2 \dots e_v})\ \delta_{\cdot e_1 e_2 \dots e_v}$$

$$=\underset{v=+\infty}{\text{Limit}}\sum_{(e_k)}\sum_{k=1}^{v} r_k.\frac{e_k^{\,l_k}}{\Pi'_k}.\frac{1}{\Pi_v}.$$

Now the complete set of partition points (0 included, 1 excluded) may be divided into Π_v / m_k sets, each set being of the type

$$x_{\cdot e_1 \dots e_{k-1}\,0\,e_{k+1}\dots e_v},\quad x_{\cdot e_1 \dots e_{k-1}\,1\,e_{k+1}\dots e_v},\ \dots\dots\ x_{\cdot e_1 \dots e_{k-1},\,m_k-1,\,e_{k+1}\dots e_v}.$$

Hence in the complete set of partition points, e_k assumes the sequence of values

$$0, 1, \dots, m_k-1$$

*Hobson, §225.

precisely Π_v / m_k times. Hence

$$\int_0^1 f(x)\,dx = \underset{v=+\infty}{\text{Limit}} \sum_{k=1}^{v} \frac{r_k}{\Pi_k} \cdot \frac{1}{\Pi_v} \cdot \sum_{(e_k)} e_k^{l_k}$$

$$= \underset{v=+\infty}{\text{Limit}} \sum_{k=1}^{v} \frac{r_k}{\Pi_k} \cdot \frac{1}{\Pi_k} \cdot \frac{\Pi_v}{m_k} \cdot \sum_{e_k=1}^{m_k-1} e_k^{l_k}$$

$$= \sum_{k=1}^{+\infty} \frac{r_k}{m_k} \cdot \frac{1^{l_k} + 2^{l_k} + \ldots + (m_k - 1)^{l_k}}{m'_1\, m'_2 \ldots m'_k}.$$

For example, let

$$1^\circ.\quad l_k = 1,\ r_k = 1,\ m_k = 10,\ m'_k = 10^2.$$

Then

$$\int_0^1 f(x)\,dx = \tfrac{1}{22}.$$

$$2^\circ.\quad l_k = 2,\ r_k = 1,\ m_k = 10,\ m'_k = 10^2.$$

Then

$$\int_0^1 f(x)\,dx = \tfrac{19}{66}.$$

Peano, *loc. cit.*, gives as the values of these integrals $\frac{5}{111}$ and $\frac{32}{111}$, respectively.

8. In conclusion, I wish to thank Professor E. H. Moore for valuable suggestions in the preparation of this paper.

NOTE.—We regret to say that the copy of this article was lost in the mails on its transit to the author. In order to avoid longer delay of this number we undertook the proof reading of the article without the manuscript. Such errors as have escaped will be pointed out in a future issue. ED. F.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Remark on solution of Problem 299, by J. M. ARNOLD, Crompton, R. I.

The following method shows *why* there is only *one* solution, a fact not clearly discernable in the published solution on page 203, Vol. XV.

Let x, y, $3y-x$ be the sides in increasing order of magnitude. Then the area must be $2y-2x$. This leads to an equation of the first degree in x, from which $x=\frac{1}{2}y+\dfrac{16y}{3y^2+16}$.

Substituting for y, the numbers 1, 2, 3, etc., we find $y=4$, $x=3$. All other integral values of y will give fractional values for x, as the second term soon becomes less than $\frac{1}{2}$, and continues to diminish as y increases.

Hence, 3, 4, 5 are the sides of the only triangle satisfying the problem.

302. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that the system of equations

$$xu+5yv=2,$$
$$xv+yu=1,$$

have no integral solution except one of the unknowns be zero.

I. Solution by E. B. ESCOTT, Ann Arbor, Mich.

Squaring and adding, and multiplying second equation by 5,

$$(xu-5yv)^2+5(xv+yu)^2=9.$$

But, since $(xu-5yv)^2+5(xv+yu)^2=(xu+5yv)^2+5(xv-yu)^2$, we must have the equation

$$X^2+5Y^2=9,$$

satisfied for two sets of values (X, Y).

The only solutions of this last equation are

$$X=\pm 2,\ \pm 3,\quad Y=\pm 1,\ 0,$$

i. e.,
$$xu-5yv=2,\quad xv+yu=1,$$
$$xu+5yv=\pm 3,\quad xv-yu=0.$$

Whence, $2xu=5$ or -1, $2xv=1$, evidently impossible.

Therefore, the only possible solutions are

$$xu-5yv=2, \quad xu+5yv=\pm 2.$$

Whence, $2xu=4$ or 0, $10yv=0$ or -4; *i. e.*, either x, u, y, or v equals zero.

Solved, similarly, by G. B. M. Zerr and V. M. Spunar.

II. Solution by O. C. CARMICHAEL, Oxford, Ala.

If the square of the first equation be added to five times the square of the second equation, we have

$$x^2u^2+5x^2v^2+25y^2v^2+5y^2u^2=9.$$

Therefore, there is no integral solution in x, y, u, v except when one of the unknowns is zero; for, if they were all positive integers, $25y^2v^2$ itself would be greater than 9.

303. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate the determinant

$$\begin{vmatrix} D_1 & x_1x_2 & x_1x_3 & \dots & x_1x_n \\ x_1x_2 & D_2 & x_2x_3 & \dots & x_2x_n \\ x_1x_3 & x_2x_3 & D_3 & \dots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & \dots & D_n \end{vmatrix}$$

Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.

$$\triangle = x_1x_2x_3\dots x_n \begin{vmatrix} D_1/x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & D_2/x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & D_3/x_3 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & x_3 & \dots & D_n/x_n \end{vmatrix}$$

$$= \prod_1^n x_1 \begin{vmatrix} (D_1-x_1^2)/x_1 & 0 & 0 & \dots & (x_n^2-D_n)/x_n \\ 0 & (D_2-x_2^2)/x_2 & 0 & \dots & (x_n^2-D_n)/x_n \\ 0 & 0 & (D_3-x_3^2)/x_3 \dots & & (x_n^2-D_n)/x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & x_3 & \dots & D_n/x_n \end{vmatrix}$$

Subtracting the last row from each row,

$$=\prod_1^n x_r . \prod_1^n \frac{D_r-x_r^2}{x_r} \begin{vmatrix} 1 & 0 & 0 & \dots & -1 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{x_1^2}{D_1-x_1^2} & \frac{x_2^2}{D_2-x_2^2} & \frac{x_3^2}{D_3-x_3^2} & \dots & \frac{D_n}{D_n-x_n^2} \end{vmatrix}$$

$$=\prod_1^n (D_r-x_r^2 \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{x_1^2}{D_1-x_1^2} & \frac{x_2^2}{D_2-x_2^2} & \frac{x_3^2}{D_3-x_3^2} & \dots & \frac{D_n}{D_n-x_n^2}+\frac{x_1^2}{D_1-x_1^2} \end{vmatrix}$$

adding last column to the first. Since the first row contains $n-1$ zeros, this reduces the determinant to one of the $(n-1)$st order. Adding last column to the first we reduce to one of the $(n-2)$nd order. Repeat this process $(n-1)$ times in all.

$$\text{Then } \triangle=\prod_1^n (D_r-x_r^2)\left[\frac{D_n}{D_n-x_n^2}+\frac{x_1^2}{D_1-x_1^2}+\frac{x_2^2}{D_2-x_2^2}+\dots\frac{x_{n-1}^2}{D_{n-1}-x_{n-1}^2}\right]$$

$$=\prod_1^n (D_r-x_r^2)\left[\frac{x_1^2}{D_1-x_1^2}+\frac{x_2^2}{D_2-x_2^2}+\dots\frac{x_{n-1}^2}{D_{n-1}-x_{n-1}^2}+\frac{x_n^2}{D_n-x_n^2}+1\right]$$

$$=\prod_1^n (D_r-x_r^2)\left[\sum_1^n \frac{x_r^2}{D_r-x_r^2}+1\right].$$

Also solved by G. B. M. Zerr, V. M. Spunar, and J. Scheffer.

304. Proposed by C. N. SCHMALL, New York City.

A policeman on a motor-cycle starts in pursuit of an automobile when the latter has a headway of $\frac{1}{2}$ a mile. A pedestrian who is $\frac{1}{4}$ mile ahead of the auto and who is walking at the rate of 5 miles an hour, notices that when the auto overtakes him the policeman is only 5-12 of a mile behind the auto, and $2\frac{1}{2}$ miles from where the officer started; he overtakes the auto. How long did the chase last?

Solution by G. B. M. ZERR, A. M., Ph. D., and the PROPOSER.

Let x=policeman's rate, y=auto's rate, and z=time for auto to overtake pedestrian.

Then $(5z+\frac{1}{4})/y=(5z+\frac{1}{4}+\frac{1}{12})/x\dots(1)$,

$2/y=(2\frac{1}{2})/x\dots(2)$.

$(1)/(2)$ gives $300z+15=240z+16$. $\therefore z=\frac{1}{60}$ hours=1 minute.

Since it takes the policeman 1 minute to gain $\frac{1}{12}$ mile, it will take him 6 minutes to overtake the auto and end the race.

In the statement of this problem, the semi-colon should be omitted after the word "started." With this omission the problem was solved as above by J, W. Clawson, V. M. Spunar, J. H. Meyer. R. D. Carmichael, A. H. Holmes, B. Kramer, J. Scheffer, J. K. Ellwood, and P. S. Berg, interpreting the problem as printed, agree in the answer being 2 hours, 36 minutes.

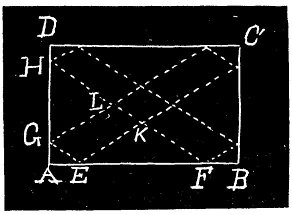
GEOMETRY.

336. Proposed by F. H. HODGE, The University of Chicago.

A man owning a rectangular field $b=300$ feet by $a=600$ feet, wishes to lay out driveways of equal width having the diagonals of the field as center lines and such that the area of the driveways shall be n/m=one-half, of the area of the field. Determine the width of the driveways.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; NELLIE WOOD, Senior Class, Drury College, Springfield, Mo.; and A. H. HOLMES, Brunswick, Me.

Put $AE=x$; then $AG=\dfrac{bx}{a}$; $\triangle EFK=\triangle GHL=\dfrac{b}{a}.\dfrac{(a-2x)^2}{4}$.



$\therefore$ space occupied by driveways$=ab-\dfrac{b}{a}(a-2x)^2$.

$\therefore ab-\dfrac{b}{a}(a-2x)^2=\dfrac{n}{m}ab$; whence

$x=\frac{1}{2}a\left(1-\sqrt{1-\dfrac{n}{m}}\right)$.

$\therefore$ breadth of driveway$=\dfrac{ab}{\sqrt{(a^2+b^2)}}\left(1-\sqrt{1-\dfrac{n}{m}}\right)$.

For $a=600$, $b-300$, $n:m=1:2$, we find breadth$=60(2\sqrt{5}-\sqrt{10})=78.572$ feet.

Also solved by B. Kramer, V. M. Spunar, G. I. Hopkins, J. H. Meyer, G. B. M. Zerr, and A. H. Bell.

337. Proposed by T. N. HILDEBRANT, The University of Chicago.

Required the locus of the vertices of the parabolae having a given focus and passing through a given point.

Solution by the PROPOSER.

Let O be the given focus and P the given point. From the properties of the parabola we see that the directrices of the parabolae passing through P will be the tangents to the circle of radius OP and center P. Hence the vertices will be the mid-points D of the perpendiculars from O to the tangents. From elementary geometry we have the triangles OCB and OAB equal, and therefore $OC=OA=x_1$ the abscissa of B the point of tangency, if we suppose O the origin of our system of coordinates. Hence $OD=\frac{1}{2}x_1$. Denote by α the angle POC. Then we also have $EPB=\alpha$. Evidently

$$x_1=OP+PA=a+a\cos\alpha=a(1+\cos\alpha);\ i.\ e.,\ OD=\tfrac{1}{2}x_1=\frac{a}{2}(1+\cos\alpha),$$

which gives as the polar equation of the locus referred to O as origin and OP as initial line,

$$\rho=\frac{a}{2}(1+\cos\alpha),$$

a cardioid, with cusp at O passing through P and having OP as line of symmetry.

Also solved by J. Scheffer, A. H. Holmes, V. M. Spunar, G. B. M. Zerr, and J. W. Clawson.

CALCULUS.

264. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The join of the center of curvature of a curve to the origin is at α to the initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d\psi^2}\right)^2\right]=\frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Denote the coordinates of the center of curvature by ξ, η. Then $\xi=x-\rho\sin\phi$, $\eta=y+\rho\cos\phi$, ρ being the radius of curvature and ϕ the angle formed by the tangent and positive x-axis.

$\therefore \tan\alpha=\eta/\xi$, and $d\alpha=\dfrac{\xi\, d\eta-\eta\, d\xi}{\xi^2+\eta^2}$.

Likewise, $\tan\psi=r\dfrac{d\theta}{dr}$, and $d\psi=\left[d\theta+r\left(\dfrac{d^2\theta}{dr^2}\right)\right]\div\sec^2\psi$, where ψ=angle included by the tangent at (r,θ), and the radius vector to the point (r,θ).

Substituting the proper values of ξ, η, $d\xi$, $d\eta$ in $d\alpha$, expressed in polar coordinates, and we have the first product $d\alpha/d\psi$.

Also, on remembering that $\left\{\begin{array}{l} p=r\sin\psi \\ \varphi=\theta+\psi\end{array}\right\}$, differentiate it twice, square every differential quotient as indicated by the proposition, add the like terms, reduce, and we have, readily,

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d\psi^2}\right)^2\right]=\frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find two curves which possess the property that the tangents TP and TQ to the inner one always makes equal angles with the tangent TT' to the outer.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In Salmon's *Conic Sections*, Sixth Edition, Articles 188-189, it is demonstrated that if through any point P of a conic section we draw tangents PT, QT to a confocal conic section, these tangents are equally inclined to the tangent at P. For every point P the outer conic through P, though confocal to the inner, has different axes.

If we wish to find a curve for all positions of P the problem assumes some difficulty.

For the ellipse, the equation to TP, TQ is

$$(a^2-h^2)(y-k)^2+2(y-k)(x-h)hk+(b^2-k^2)(x-h)^2=0,$$

where (h, k) are the coordinates of T.

$$2hk(y-k)+\{b^2-a^2+h^2-k^2\pm\sqrt{[(b^2-a^2+h^2-k^2)^2+4h^2k^2]}\}(x-h)=0,$$

gives the two lines making equal angles with TP, TQ.

The envelope of the line formed by using the plus sign subject to the condition obtained by using the minus sign gives the required curve.

If $a=b$, the ellipse becomes a circle. Then TT' is

$$ky+hx=h^2+k^2\ldots(1).$$

$hy=kx$ is the perpendicular to (1).

The envelope of (1) subject to the condition $hy=kx$,is $x^2+y^2=0$, or the center of the given circle.

266. Proposed by C. N. SCHMALL, New York City.

Show that the nth derivative of the fraction u/v can be expressed in the form of a determinant, u and v being functions of x.

Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

As $u=\phi(x)$, $v=\psi(x)$, hence, $F(x)=\phi(x)/\psi(x)=u/v$, and therefore,

$$F'(x)=\frac{vu'-uv'}{v^2}=\frac{1}{v^2}\begin{vmatrix} v & v' \\ u & u' \end{vmatrix}=\frac{u_1}{v_1}=\frac{\phi_1(x)}{\psi_1(x)}.$$

Likewise, $F''(x)=\dfrac{1}{v_1^2}\begin{vmatrix} v_1 & v'_1 \\ u_1 & u'_1 \end{vmatrix}$; $F'''(x)=\dfrac{1}{v_2^2}\begin{vmatrix} v_2 & v'_2 \\ u_2 & u'_2 \end{vmatrix}$; ...

$$\therefore F^{(n)}(x)=\frac{1}{v^2_{n-1}}\begin{vmatrix} v_{n-1} & v'_{n-1} \\ u_{n-1} & u'_{n-1} \end{vmatrix},\ \text{where}\ v_\lambda=v^{2^\lambda},\quad v'_\lambda=2^\lambda v^{2^\lambda-1}v';$$

$$u_\lambda=\left|\begin{matrix} v_{\lambda-1} & v'_{\lambda-1} \\ u_{\lambda-1} & u'_{\lambda-1}\end{matrix}\right|, \text{ and } u'_\lambda=\left|\begin{matrix} v_{\lambda-1} & v''_{\lambda-1} \\ u_{\lambda-1} & u''_{\lambda-1}\end{matrix}\right|\cdots$$

Also solved by G. B. M. Zerr.

267. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

A point within an ellipse, upon a normal making an angle λ with the major axis, is arbitrarily chosen. With this point as pole, and the line through it parallel to the major axis as polar axis, the equation of the ellipse is, $A\cos^4\theta+B\cos^3\theta+C\cos^2\theta+D\cos\theta+E=0$, where the coefficients are functions of λ, of the radius vector ρ, and of the distance along the normal to the pole, ρ_1. Evidently for $\rho=\rho_1$, a solution is $\cos\theta=\cos\lambda$. Required the multiplicity of this solution for any values of ρ_1, $[\lambda\neq0,\ \rho_1\neq0]$.

Solution by the PROPOSER.

I. *Analytical Solution.* The normal to $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1$ at (x_0, y_0) is, denoting its slope by $\tan\lambda$, $\dfrac{x_0-\xi}{\cos\lambda}=\dfrac{y_0-\eta}{\sin\lambda}=r$, where r is the distance from (x_0, y_0) to any point (ξ, η), being positive for interior points. Now, $\tan\lambda=-\dfrac{dx_0}{dy_0}=\dfrac{x_0}{x_0(1-e^2)}$, whence $x_0^2+x_0^2(1-e^2)\tan^2\lambda=a^2$, or

(1) $$x_0=\mu\cos\lambda,\ y_0=\mu\sin\lambda(1-e^2),\ \mu=\frac{a}{\sqrt{(1-e^2\sin^2\lambda)}}.$$

And, if the pole (x_1, y_1):

(2) $$x_1=x_0-\rho_1\cos\lambda,\ y_1=y_0-\rho_1\sin\lambda.$$

Transfer the origin to (x_1, y_1) and introduce polar coordinates by $x=x_1+\rho\cos\theta$, $y=y_1+\rho\sin\theta$; the ellipse becomes

(3) $$(1-e^2)(x_1+\rho\cos\theta)^2+(y_1+\rho\sin\theta)^2=a^2(1-e^2).$$

Collect, and eliminate $\sin\theta$:

(4) $$(\alpha\cos^2\theta+\beta\cos\theta+\gamma)^2=(-\delta\sin\theta)^2=\delta^2(1-\cos^2\theta),$$

(5) $$\text{where } \alpha=-e^2\rho^2,\ \beta=2x_1\rho(1-e^2),\ \delta=2y_1\rho,$$
$$\text{and } \gamma=\rho^2+(x_1^2-a^2)(1-e^2)+y_1^2.$$

The re-arrangement of (4) in descending powers of $\cos\theta$ gives the form of equation used in stating the problem; the above form will, however, suffice.

When $\rho=\rho_1$ let the coefficients be denoted by $\alpha_1, \ldots, \delta_1$; and let $F(\cos\theta)\equiv(\alpha_1\cos^2\theta+\beta_1\cos\theta+\gamma_1)^2-\delta_1^2(1-\cos^2\theta)$. Now, from the derivative of (4), it is obvious that $\cos\theta=\cos\lambda$ is a solution when $\rho=\rho_1$; *i. e.*, $F(\cos\lambda)=0$. The multiplicity of this solution will be shown by the derivatives. Differentiating with respect to $\cos\theta$, and substituting $\cos\lambda$:

$$\begin{aligned}(6)\qquad F'(\cos\lambda)&\equiv 2(\alpha_1\cos^2\lambda+\beta_1\cos\lambda+\gamma)(2\alpha_1\cos\lambda+\beta_1)+2\delta_1^2\cos\lambda\\ &=-2\delta_1\sin\lambda(2\alpha_1\cos\lambda+\beta_1)+2\delta_1^2\cos\lambda\end{aligned}$$

because of (4). Substituting by (5), (2) and (1):

$$F'(\cos\lambda)=4\delta_1\rho_1\sin\lambda\cos\lambda[e^2\rho_1-(\mu-\rho_1)(1-e^2)+\mu(1-e^2)-\rho_1]\equiv 0.$$

$$\begin{aligned}(7)\quad \text{Again.}\quad F''(\cos\lambda)&\equiv 4\alpha_1(\alpha_1\cos^2\lambda+\beta_1\cos\lambda+\gamma_1)+2(2\alpha_1\cos\lambda+\beta_1)^2+2\delta_1^2\\ &\equiv -4\alpha_1\delta_1\sin\lambda+2\delta_1^2\cot^2\lambda+2\delta_1^2,\end{aligned}$$

because of (4) and (6). Substituting by (5), (2) and (1):

$$(8)\qquad F''(\cos\lambda)=\frac{4\rho_1^2y_1}{\sin\lambda}[\mu(1-e^2)-\rho_1(1-e^2\sin^2\gamma)].$$

This vanishes if, and only if, $y_1=0$ or $\rho_1=\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$.

$$(9)\qquad \text{Again, } F'''(\cos\lambda)\equiv 12\alpha_1[2\alpha_1\cos\lambda+\beta_1]=\frac{12\alpha_1\delta_1\cos\lambda}{\sin\lambda}.$$

This vanishes if, and only if, $y_1=0$ or $\cos\lambda=0$.

Geometrically, these results show that, with the pole at any point on the normal, the solution $\cos\theta=\cos\lambda$ is of order at least two; with the pole at the center of curvature for the given normal, the solution is of order three; with the pole on the major axis, of order four.

[That $\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$ is the radius of curvature is seen thus: the usual expression, $R=\dfrac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, reduces immediately by using (1) and $b^2=a^2(1-e^2)$ to $\dfrac{\mu^3(1-e^2)}{a^2}$ or $\dfrac{\mu(1-e^2)}{1-e^2\sin^2\lambda}$.]

The vanishing of $F'''(\cos\lambda)$ when $\cos\lambda=0$ is unimportant, since $F''(\cos\lambda)\neq 0$ when $\cos\lambda=0$.

II. *Geometrical Solution.* Geometrically, $\rho=\rho_1$ is the equation of a circle, of radius ρ_1, with its center at the pole. The order of the solution

$\cos\theta=\cos\lambda$ of the given equation is at least as great as the number of coincident points in which the circle cuts the ellipse at the given point. [How the order may be greater than this number will appear shortly.]

The pole being any point whatever on the normal, the circle is tangent to the ellipse, since the two curves have a common normal. There are consequently at least two coincident points of intersection.

If the pole be at the center of curvature for the given normal, the circle is the osculating circle, which cuts the ellipse in at least three coincident points.

If the pole be on the major axis, then it follows from the symmetry that the circle is tangent to the ellipse at two points, where $\theta=\lambda$ and where $\theta=-\lambda$. But since $\cos(-\lambda)=\cos\lambda$, it follows that for all four intersections $\cos\theta=\cos\lambda$, so that this solution is of multiplicity four.

MECHANICS.

220. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Four particles A, B, C, D, lie on a smooth table at the corners of a rhombus. AB, BC, CD, DA are light inextensible strings connecting the particles. The angle at A is acute. A blow is given to A along the diagonal, away from C. Find the ratio of the initial velocity of C to that of A.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let v=velocity of A. Then since the table and particles are smooth the component of v along the direction BA is $y=v\cos\frac{1}{2}A$. The component of y in the direction CB is $z=y\cos A=v\cos A\cos\frac{1}{2}A$. The component of z in the direction CA is $u=z\cos\frac{1}{2}A$.

$\therefore u=v\cos A\cos^2\frac{1}{2}A$. But u is the initial velocity of C.

$\therefore v:u=1:\cos A\cos^2\frac{1}{2}A$.

As $\angle A$ decreases every instant, u increases every instant until $\angle A=0°$, $u=v$.

221. Proposed by W. J. GREENSTREET, Stroud, England.

Two smooth intersecting planes are each at 45° to the horizon. Between them lies a cylinder of elliptic cross section. Find the position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let P, Q be the points of contact of the cylinder with the planes, intersecting at an angle of 90°. Let the normals at P, Q intersect in O and let C be the center of an elliptic section in the same plane as OP, OQ. Then either C and O coincide or CO is vertical. If they coincide the axes of the elliptic section are parallel to the planes. This gives one position of equilibrium. When OC is vertical, OC makes an angle of 45° with both planes.

From the force diagram the reactions at P, Q are equal, hence either the major axis is vertical or the minor axis is vertical, giving the other two positions of equilibrium.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

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156. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for a, b, c, d, and e in the equation, $a^2+b^2+c^2+d^2=e^2$.

I. Solution by the PROPOSER.

Take $a=32(40^2-9^2)$; $b=64(40\times9)$; $c=24(40^2+9^2)$; $d=9(40^2+9^2)$. Then $a^2=32^2\times1519^2$; $b^2=32^2\times720^2$; $c^2=24^2\times1681^2$; $d^2=81\times1681^2$. And $e^3=32^2[1519^2+720^2+1681^2(24^2+81)]$.

$\therefore a=48608$; $b=23040$; $c=40344$; $d=15129$; $e=1681$.

In the above problem it was intended by the author that e^2 should be e^3. ED. F.

II. Solution by ARTEMAS MARTIN, LL. D., Coast Survey Office, Washington, D. C.

In the well-known identity

$$(x-y)^2+4xy=(x+y)^2, \tag{1}$$

x and y may have any values whatever, and we can, therefore, assume $x=u+v+w$; then (1) becomes

$$(u+v+w-y)^2+4y(u+v+w)=(u+v+w+y)^2. \tag{2}$$

Now take $u=p^2$, $v=q^2$, $w=r^2$, $y=s^2$; then (2) becomes

$$(p^2+q^2+r^2-s^2)^2+(2ps)^2+(2qs)^2+(2rs)^2=(p^2+q^2+r^2+s^2)^2, \quad (3);$$

and p, q, r, s may be any values chosen at pleasure.

Therefore we may take $a=p^2+q^2+r^2+s^2$, $b=2ps$, $c=2qs$, $d=2rs$, $e=p^2+q^2+r^2+s^2$, or in any other order we please.

Examples. 1. Take $p=1$, $q=2$, $r=3$, $s=1$; then $2^2+4^2+6^2+13^2=15^2$.

2. Take $p=1$, $q=2$, $r=3$, $s=4$; then, after dividing through by 2^2 and discarding the negative sign, $1^2+4^2+8^2+12^2=15^2$.

3. Take $p=2$, $q=3$, $r=4$, $s=5$; then, after dividing through by 2^2, $2^2+10^2+15^2+20^2=27^2$.

See *Mathematical Magazine*, Vol. II, No. 5, pp. 69-76; No. 6, pp. 89-96; No. 8, pp. 137-140; and No. 11, pp. 209-220, for various methods of finding many sets of square numbers whose sum is a square.

III. Solution by **ARTEMAS MARTIN, LL. D.**, Coast Survey Office, Washington, D. C.

Let $a=x+p$, $b=x-p$, $c=x+q$, $d=x-q$, $e=2x+n$; then by substitution and reduction, $4x^2+2p^2+2q^2$, from which

$$x=\frac{2p^2+2q^2-n^2}{4n}.$$

Substituting the value of x and rejecting the common denominator $4n$, we may take

$$a=2p^2+2q^2-n^2+4np,$$
$$b=2p^2+2q^2-n^2-4np,$$
$$c=2p^2+2q^2-n^2+4nq,$$
$$d=2p^2+2q^2-n^2-4nq,$$
$$e=4p^2+4q^2+2n^2.$$

Examples. 1. Take $p=3$, $q=2$, $n=1$; then we have $13^2+17^2+33^2+37^2=54^2$.

2. Take $p=1$, $q=2$, $n=3$, and we have $11^2+13^2+23^2+25^2=38^2$.

3. Take $p=3$, $q=4$, $n=2$, and we get $7^2+11^2+35^2+39^2=54^2$.

Also solved by V. M. Spunar, S. E. Corey, G. B. M. Zerr, and J. Scheffer.

NOTES AND NEWS.

This first number of the sixteenth volume of the MONTHLY has been delayed for various reasons, including the culmination of plans whereby its future publication is to be under the joint auspices of The University of Chicago and the University of Illinois. In accordance with this plan, Professor G. A. Miller, of the University of Illinois, and Professor H. E. Slaught, of the University of Chicago, will share jointly the editorial responsibility of articles. Professor B. F. Finkel, of Drury College, will continue in charge of the problem department and have general management of the MONTHLY, as heretofore, and Professor L. E. Dickson, of The University of Chicago, whose editorial cooperation has been of the highest value, will retire from active service. It is believed that the union of these two universities in the interests of the MONTHLY will result in the further development of its usefulness and in the extension of its influence in its particular field, namely, the realm of college mathematics as distinguished, on the one hand, from that of the secondary schools, and on the other hand, from that of the graduate schools.

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SOME NOTES ON VECTOR ANALYSIS.*

By ARTHUR C. LUNN, The University of Chicago.

1. THE PARAMETRIC REPRESENTATION OF A DYADIC OF ROTATION.

If a rigid body undergo any displacement or change of orientation, keeping one point fixed, the rectangular coordinates of any point after the motion, referred to the fixed point as origin, are expressible in terms of the coordinates of the same point before the motion by the equations

$$
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z,\\
y' &= a_{21}x + a_{22}y + a_{23}z,\\
z' &= a_{31}x + a_{32}y + a_{33}z,
\end{aligned}
\tag{1}
$$

where the coefficients a_{ij} constitute an orthogonal matrix or dyadic, being subject to the conditions

$$
\begin{aligned}
a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1, & a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0,\\
a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1, & a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} &= 0,\\
a_{13}^2 + a_{23}^2 + a_{32}^2 &= 1, & a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0,
\end{aligned}
\tag{2}
$$

which are obvious geometrically from the interpretation of the a's, and of these combinations of them, as cosines of angles formed by various pairs of lines; or algebraically from the fact that the relation $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$ must be an identity in x, y, z. These relations are independent and sufficient, so that three of the coefficients are arbitrary; or, more generally, the nine coefficients may be considered as functions of three arbitrary parameters.

Euler showed that any such displacement could be produced by a single rotation through a certain angle about a determinate axis, and therefore that the coefficients could be expressed in terms of the angle of rotation t

*For the notation and theorems of vector analysis here used, see for example: Gibbs-Wilson, *Vector Analysis*, to which reference is made in following foot-notes.

and the direction-cosines λ, μ, ν of the axis of rotation, these being of course subject to the relation $\lambda^2+\mu^2+\nu^2=1$. The resulting values of the coefficients, arranged as in (1) are

$$(3)\qquad \begin{array}{lll} \lambda^2+(1-\lambda^2)\cos t, & \lambda\,\mu(1-\cos t)-\nu\,\sin t, & \lambda\,\nu(1-\cos t)+\mu\,\sin t, \\ \lambda\,\mu(1-\cos t)+\nu\,\sin t, & \mu^2+(1-\mu^2)\cos t, & \mu\,\nu(1-\cos t)-\lambda\,\sin t, \\ \lambda\,\nu(1-\cos t)-\mu\,\sin t, & \mu\,\nu(1-\cos t)+\lambda\,\sin t, & \nu^2+(1-\nu^2)\cos t. \end{array}$$

These were proved by means of geometric construction by Rodrigues,* who also by the substitution

$$(4)\qquad \lambda\tan\frac{t}{2}=\alpha,\quad \mu\tan\frac{t}{2}=\beta,\quad \nu\tan\frac{t}{2}=\lambda,$$

reduced them to the form

$$(5)\qquad \begin{array}{lll} \dfrac{1+\alpha^2-\beta^2-\gamma^2}{1+\tau^2}, & \dfrac{2\,\alpha\,\beta-2\,\gamma}{1+\tau^2}, & \dfrac{2\,\alpha\,\gamma+2\,\beta}{1+\tau^2}, \\ \dfrac{2\,\alpha\,\beta+2\,\gamma}{1+\tau^2}, & \dfrac{1+\beta^2-\gamma^2-\alpha^2}{1+\tau^2}, & \dfrac{2\,\beta\,\gamma-2\,\alpha}{1+\tau^2}, \\ \dfrac{2\,\alpha\,\gamma-2\,\beta}{1+\tau^2}, & \dfrac{2\,\beta\,\gamma+2\,\alpha}{1+\tau^2}, & \dfrac{1+\gamma^2-\alpha^2-\beta^2}{1+\tau^2}, \end{array}$$

(where $\tau=\tan\frac{1}{2}t$, $\tau^2=\alpha^2+\beta^2+\gamma^2$) which shows explicitly their dependence on three arbitrary parameters (α, β, γ), these being geometrically the components of a vector in the direction of the axis of rotation whose length is equal to the tangent of half the angle of rotation.

The following deduction leads to the same expressions by means of integration in series of the vector differential equation which gives the law of distribution of velocity at various points of a rotating rigid body.

Let **u** be the constant vector of angular velocity, in the direction of the axis, and **r** the vector from the origin to any point, then as a vector the linear velocity at that point is†

$$(6)\qquad \mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{u}\times\mathbf{r},$$

where t stands for the time but may conveniently be identified with the angle of rotation by taking the angular velocity as numerically equal to unity, so that **u** is the unit-vector whose components are (λ, μ, ν). Then

*Liouville's *Journal de Mathematique*, Ser. 1, t. V, p. 380.

†*Vector Analysis*. p. 98.

$$\frac{d^2\mathbf{r}}{dt^2}=\mathbf{u}\times\frac{d\mathbf{r}}{dt}=\mathbf{u}\times(\mathbf{u}\times\mathbf{r})=(\mathbf{u}.\mathbf{r})\mathbf{u}-\mathbf{r},$$

(7)

$$\frac{d^3\mathbf{r}}{dt^3}=\left(\mathbf{u}.\frac{d\mathbf{r}}{dt}\right)\mathbf{u}-\frac{d\mathbf{r}}{dt}=-\frac{d\mathbf{r}}{dt},$$

since $\mathbf{u}.\frac{d\mathbf{r}}{dt}=\mathbf{u}.(\mathbf{u}\times\mathbf{r})=0$; and by induction

(8) $$\frac{d^{n+2}\mathbf{r}}{dt^{n+2}}=-\frac{d^n\mathbf{r}}{dt^n}$$

for every $n>0$. Hence Maclaurin's series

$$\mathbf{r}_t=\mathbf{r}_0+\frac{d\mathbf{r}}{dt}.t+\frac{d^2\mathbf{r}}{dt^2}.\frac{t^2}{2\,!}+\ldots$$

reduces to

$$\mathbf{r}_t=\mathbf{r}_0+\Big[(\mathbf{u}.\mathbf{r}_0)\mathbf{u}-\mathbf{r}_0\Big]\left[\frac{t^2}{2\,!}-\frac{t^4}{4\,!}+\ldots\right]$$

$$+\Big[\mathbf{u}\times\mathbf{r}_0\Big]\left[\frac{t}{1\,!}-\frac{t^3}{3\,!}+\ldots\right]$$

where the series admit immediate interpretation, giving

(9) $$\mathbf{r}_t=\mathbf{r}_0+[(\mathbf{u}.\mathbf{r}_0)\mathbf{u}-\mathbf{r}_0][1-\cos t]+[\mathbf{u}\times\mathbf{r}_0]\sin t.$$

Here $\mathbf{r}_0$, $\mathbf{r}_t$ are the vectors from the origin to the same point of the body before and after the rotation, respectively, their components being (x, y, z), (x', y', z'); also $\mathbf{u}.\mathbf{r}_0=\lambda\, x+\mu\, y+\nu\, z$, and the components of $\mathbf{u}\times\mathbf{r}_0$ are $(\mu\, z-\nu\, y,\ \nu\, x-\lambda\, z,\ \lambda\, y-\mu\, x)$. Thus equation (9) is equivalent to the Cartesian equations (1) with coefficients expressed as in (3).

2. A Representation of a Solenoidal Vector.

In the general theory of vector functions of a point in space* it is known that a necessary and sufficient condition that a vector have a vanishing curl is that it admit of representation as a potential vector, or one whose components are the partial derivatives of a single scalar function. Symbolically this means that if $\mathbf{v}=\nabla S$, then curl $\mathbf{v}=0$, and conversely, if curl $\mathbf{v}=0$ then there exists a scalar function S such that $\mathbf{v}=\nabla S$. The present note is concerned with an analogous theorem for a solenoidal vector, or one whose divergence vanishes.

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* *Vector Analysis*, Chap. III.

It is known, as readily proved by direct differentiation, that if a vector function be the vector product of two potential vectors it is solenoidal; or symbolically, if $\mathbf{v} = \nabla U \times \nabla V$, then $\operatorname{div} \mathbf{v} = 0$. The following proof establishes the converse theorem, that if $\operatorname{div} \mathbf{v} = 0$ then there exist in general two scalar functions U, V, such that $\mathbf{v} = \nabla U \times \nabla V$.

Let $\mathbf{v}$ be at first any vector function, and u, v two independent integrals of the linear partial differential equation in three independent variables

(1) $$\mathbf{v} . \nabla u = 0,$$

which means geometrically that u, v are two functions such that ∇u and ∇v are not collinear, but are both everywhere perpendicular to $\mathbf{v}$. Hence $\mathbf{v}$ can be written

(2) $$\mathbf{v} = w \nabla u \times \nabla v,$$

where w is some scalar function. Then

$$\operatorname{div} \mathbf{v} = w \operatorname{div}(\nabla u \times \nabla v) + \nabla w . (\nabla u \times \nabla v)$$

where the first term vanishes identically so that

(3) $$\operatorname{div} \mathbf{v} = [\nabla u, \ \nabla v, \ \nabla w],$$

this being analytically the Jacobian or functional determinant of the functions u, v, w, with respect to the variables x, y, z, the Cartesian coordinates.

If now $\mathbf{v}$ be solenoidal, or this Jacobian vanish, a well known theorem shows that w must be a function of u, v; so that

(4) $$\mathbf{v} = f(u, v) \nabla u \times \nabla v.$$

Now let U, V be two functions of u, v; then

(5) $$\nabla U \times \nabla V = J \nabla u \times \nabla v$$

where

(6) $$J = \begin{vmatrix} \frac{\partial U}{\partial u} & \frac{\partial U}{\partial v} \\ \frac{\partial V}{\partial u} & \frac{\partial V}{\partial v} \end{vmatrix}.$$

It is clearly possible to choose U, V, which are also integrals of (1), so that J, which is a function of u, v, shall be equal to $f(u, v)$; for instance, by taking $U = u$, $V = \int f(u, v) dv$. With any such choice then $\mathbf{v}$ takes the form

(7) $$\mathbf{v} = \nabla U \times \nabla V.$$

A necessary and sufficient condition that a vector function be solenoidal is that it admit of representation as the vector product of two potential vectors.

It should be noted that, when $\mathbf{v}$ is given, U and V are not uniquely determined, so that further conditions may perhaps conveniently be imposed upon them in special cases.

THE POSSIBLE ABSTRACT GROUPS OF THE TEN ORDERS 1909 — 1919.

By DR. G. A. MILLER, University of Illinois.

The real essence of fundamental theorems is frequently exhibited most forcibly by means of illustrative examples, especially when these examples have other elements of interest. The determination of all the possible abstract groups whose orders are equal to the numbers of the ten years 1909—1919 offers numerous instructive illustrations of important theorems, and exhibits some properties of these numbers which are at least of temporary interest.

Since 1909=23.83 is the product of two distinct primes such that the larger diminished by unity is not divisible by the smaller, it results that *the cyclic group of order* 1909 *is the only possible group of this order.* That is, there is only one group whose order is equal to the number of the present year. On the contrary, 1910=2.5.191 is the product of three distinct primes and hence every group of order 1910 contains an invariant subgroup of order 191 and also an invariant subgroup of order 955.* The latter may be either cyclic or non-cyclic, since 191—1 is divisible by 5.

As the group of isomorphisms of the cylic group of order 955 involves three operators of order 2 and the identity, there are four groups of order 1910 which involve a cyclic subgroup of order 955. The group of isomorphisms of the non-cyclic group of order 955 is the holomorph of the group of order 191 and hence it contains only one set of conjugate operators of order 2. When the invariant subgroup of order 955 is non-cyclic an operator of order 2 in the entire group must either transform each operator of this invari nt subgroup into itself or it must transform these operators according to an operator of order 2 in the group of isomorphisms. Hence there are two groups of order 1910 involving a non-cyclic subgroup of the order 955 and *there are exactly six distinct groups of order* 1910; *four of them involve a cyclic subgroup of order* 955 *while the remaining two do not have this property.*

*Cf. Burnside's *Theory of Groups of Finite Order*, 1897, p. 353.

The number $1911=3.7^2.13$ involves four prime factors but these factors are of such a nature that it is comparatively easy to determine all the possible groups of this order. Since 1 is the only divisor of 1911 which is of the form $13k+1$ it results from Sylow's theorem that a group of order 1911 can involve only one subgroup of order 13, and for similar reasons it can involve only one subgroup of order 49. Hence every group of order 1911 contains an invariant abelian subgroup of order 637. There are exactly five groups of order 1911 containing a cyclic subgroup of order 637 since the group of isomorphisms of this cyclic group involves four subgroups of order 3 and the identity. It is not difficult to see that there are ten groups of order 1911 containing the non-cyclic group of order 637, and hence *there are exactly fifteen distinct groups of order* 1911; *two of them are abelian, while the remaining thirteen are non-abelian.*

Every group of order $1912=3^3.239$ contains an invariant subgroup of order 239. If a subgroup of order 8 is also invariant the entire group is the direct product of the group of order 239 and one of the five groups of order 8. Hence there are exactly five groups of order 1912 such that each involves only one subgroup of order 8. If the subgroups of order 8 are not invariant there must be 239 such subgroups having four common operators, since the group of isomorphisms of the group of order 239 does not involve a subgroup of order 4. There are four possible groups in which these common operators form a cyclic group and three in which they form a non-cyclic group. Hence *the total number of abstract groups of order* 1912 *is* 12. Since 1913 is a prime *there is only one group of order* 1913.

The next number, $1914=2.3.11.29$, is the product of distinct primes and hence every group of this order involves an invariant subgroup of each of the orders 29, 29.11, 29.13.3. Each of these invariant subgroups is cyclic* and an operator of order 2 may transform the operators of orders 29, 11, and 3, either into themselves or into their inverses. Hence an operator of order 2 may transform the cyclic subgroup of order 957 in eight distinct ways and *there are exactly eight groups of order* 1914. Since $1915=5.383$ and $383-1$ is not divisible by 5 *there is only one group of order* 1915. *There are only four groups of order* $1916=2^2.479$ since $479-1$ is not divisible by 4.

A group of order $1917=3^3.71$ must be the direct product of the subgroups of order 71 and 27 respectively since $71\equiv 2$ mod. 3. Hence there are as many distinct groups of order 1917 as there are distinct groups of order 27; that is, *the number of abstract groups of order* 1917 *is* 5. A group of order $1918=2.7.137$ contains an invariant cyclic subgroup of order 7.137 and hence *there are four groups of order* 1918. It is clear that *there is only one group of order* $1919=19.101$. The next number, $1920=2^7.3.5$, lies outside the ten numbers under consideration but it may be remarked that it would be very much more difficult to determine the groups of this one order than it was to determine the groups of the ten orders under consideration. The

***Bulletin of the American Mathematical Society*, Vol. 5 (1899), p. 235.

year 1920 is so far ahead that it is to be hoped that before it arrives group theory may have made sufficient progress to determine all the groups of this order by means of general theorems. It need scarcely be added that most of the results given above are special cases of general theorems which were not mentioned in every case, since the direct proofs are so evident.

NOTE ON THE GENERAL QUARTIC.

By M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

The general quartic equation

(1) $$\phi(x) \equiv a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

where the coefficients are all real, can be reduced to the form

(2) $$\phi(x) \equiv ax^4+bx^2+c=0,$$

by the following method.

Suppose the quartic resolved into the factors:

$$(a_1x^2+b_1x+c_1) \text{ and } (a_2x^2+b_2x+c_2).$$

Then effect the rational bilinear transformation $x=\dfrac{\alpha y+\beta}{y+1}$, obtaining

$$\phi(x)=\left[\frac{a_1(\alpha y+\beta)^2}{(y+1)^2}+\frac{b_1(\alpha y+\beta)}{y+1}+c_1\right]\left[\frac{a_2(\alpha y+\beta)^2}{(y+1)^2}+\frac{b_2(\alpha y+\beta)}{y+1}+c_2\right]=0.$$

In this expression α and β may be chosen so that the coefficients of the first powers of y shall be zero, after clearing of fractions.

The values of α and β fulfilling this condition are easily found to depend upon

(3) $$\alpha\beta=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},$$ (4) $$\alpha+\beta=2\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}.$$

That α and β are real, is determined from the equation

$$(5)\qquad t^2+2\frac{(a_2c_1-a_1c_2)}{(a_1b_2-a_2b_1)}t+\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}=0,$$

by means of the discriminant relation

$$(6)\qquad \triangle\equiv\left[\frac{2\,(a_2c_1-a_1c_2)}{(a_1b_2-a_2b_1)}\right]^2-4\left[\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right]$$

$$=\frac{4\,[(a_2c_1-a_1c_2)^2-(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)]}{(a_1b_2-a_2b_1)^2}.$$

If v_1, w_1, and v_2, w_2 are the two pairs of roots of the quartic, we may assume, without loss of generality, that v_1 is greater than either v_2 or w_2 and w_1 is greater than either v_2 or w_2. From the two quadratic factors whose roots are, respectively, v_1, w_1, and v_2, w_2, we obtain

$$(7)\qquad -\frac{b_1}{a_1}=v_1+w_1,\quad \frac{c_1}{a_1}=v_1w_1,\quad -\frac{b_2}{a_2}=v_2+w_2,\ \text{and}\ \frac{c_2}{a_2}=v_2w_2,$$

and these values substituted in the discriminant give for the determinations of its signs,

$$(8)\qquad a_1^2a_2^2(v_1-v_2)(v_1-w_2)(w_1-v_2)(w_1-w_2)$$

$$=(a_2c_1-a_1c_2)^2-(b_1c_2-b_2c_1)(a_1b_2-a_2b_1).$$

If all the roots are real it is evident from (8) that the discriminant is positive. Likewise, if two roots are real and two imaginary, and also if all four roots are imaginary, the discriminant is seen to be positive, the conjugate pairs being, v_1, w_1, and v_2, w_2. Hence, (5) has real roots, and consequently α and β are real.

In the exceptional case where $\frac{b_1}{a_1}=\frac{b_2}{a_2}$, the substitution

$$x=y-\frac{b_1}{2a_1}=y-\frac{b_2}{2a_2},$$

gives directly $\phi(x)=A_1y^4+A_2y^2+A_3$, where A_1, A_2, A_3 are real constants.

Finally, from (3), (4) and (7),

$$(9)\qquad \alpha\beta=\frac{(v_2+w_2)v_1w_1-v_2w_2(v_1+w_1)}{(v_1+w_1)-(v_2+w_2)},$$

(10) $$\text{and } \alpha+\beta=\frac{2v_1w_1-2v_2w_2}{(v_1+w_1)-(v_2+w_2)}.$$

From these equations it is easily seen that

$$2\alpha\beta-(\alpha+\beta)(v_1+w_1)=-2v_1w_1,$$
$$\text{and } 2\alpha\beta-(\alpha+\beta)(v_2+w_2)=-2v_2w_2.$$

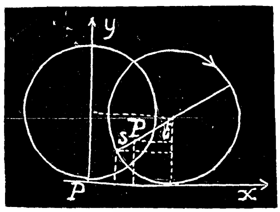
But these are the characteristic equations for harmonic pairs. Therefore, the pair α, β is harmonic with respect to the pairs v_1, w_1 and v_2, w_2.

ANOTHER WAY TO GENERATE THE CYCLOID.

By H. SCHAPPER, Fayetteville, Ark.

Consider a point describing a simple harmonic motion along the diameter of a uniformly rolling circle, and let both motions begin simultaneously. If r be the radius of the circle, and ϵ the angle of rolling, and if a complete unrolling of the circle correspond to a full period of the simple harmonic motion, then we get

$$s=r(1-\cos\epsilon)$$



as the equation for the simple harmonic motion.
From the figure we see that

$$x=r(\epsilon-\sin\epsilon)+r(1-\cos\epsilon)\sin\epsilon,$$
$$y=r(1-\cos\epsilon)+r(1-\cos\epsilon)\cos\epsilon.$$

After an easy reduction we get

$$x=r(\epsilon-\tfrac{1}{2}\sin 2\epsilon),\quad y=r\sin^2\epsilon.$$

These equations may be written in the following way, or

$$x=\frac{r}{2}(2\epsilon-\sin 2\epsilon),\quad y=\frac{r}{2}(1-\cos 2\epsilon),$$

or

$$x=R(\phi-\sin\phi),\quad y=R(1-\cos\phi),$$

which is the parametric form of the equations of the cycloid. The cycloid thus generated is the same as if described by a simple rolling of a circle of radius $\frac{1}{2}r$. From

$$\frac{dy}{dx}=\frac{\sin 2\varepsilon}{(1-\cos 2\varepsilon)}=\cot\varepsilon=\tan(\tfrac{1}{2}\pi-\varepsilon),$$

it follows that the diameter $2r$ of the circle is always tangent to the curve, so that we get the points of the curve as well as their corresponding tangents at the same time, which is rather an advantage. The construction of the cycloid is thus simplified. Besides, it is of interest that the definition of the cycloid is thus given in a different way from the one commonly employed.

DEPARTMENTS.

PROBLEMS FOR SOLUTION.

ALGEBRA.

305. Proposed by S. A. COREY, Hiteman, Iowa.

Prove or disprove, that $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2+4}=\frac{\pi}{4}-\frac{\pi}{8}\left(\frac{\cosh\pi}{\sinh\pi}\right)$.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Taking the logarithm of the well known formula

$$\cos\tfrac{1}{2}\pi x=\left(1-\frac{x^2}{1}\right)\left(1-\frac{x^2}{9}\right)\left(1-\frac{x^2}{25}\right)\ldots$$

and differentiating, we have

$$\tfrac{1}{2}\pi\tan\tfrac{1}{2}\pi x=\frac{2x}{1-x^2}+\frac{2x}{9-x^2}+\frac{2x}{25-x^2}+\ldots$$

Putting $x\sqrt{-1}$ in place of x, we have

$$\tfrac{1}{2}\pi\tan(\tfrac{1}{2}\pi x\sqrt{-1})=2x\sqrt{-1}\left[\frac{1}{1+x^2}+\frac{1}{9+x^2}+\frac{1}{25+x^2}+\ldots\right]$$

Putting $x=2$, we have

$$\tfrac{1}{2}\pi\tan(\pi\sqrt{-1})=4\sqrt{-1}\left[\frac{1}{1+4}+\frac{1}{9+4}+\frac{1}{25+4}+\dots\right]$$

But, $\tan(\pi\sqrt{-1})=\dfrac{\tanh\pi}{\sqrt{-1}}$; $\therefore\dfrac{1}{1+4}+\dfrac{1}{9+4}+\dfrac{1}{25+4}+\dots=\tfrac{1}{8}\pi\tanh\pi$.

Also solved by G. B. M. Zerr, and V. M. Spunar.

306. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Muir gives the following problem:

Prove: $$\begin{vmatrix} 1 & a & a & a^2 \\ 1 & b & b & b^2 \\ 1 & c & c' & cc' \\ 1 & d & d' & dd' \end{vmatrix} = (a-b)\begin{vmatrix} 1 & ab & a+b \\ 1 & cd' & c+d' \\ 1 & c'd & c'+d \end{vmatrix}$$

which, of course, can be solved by finding the terms of both determinants. Is there any method of changing from one form to the other which is direct?

Solution by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.

Subtracting the first row from each one of the others:

$$\begin{vmatrix} 1 & a & a & a^2 \\ 0 & b-a & b-a & b^2-a^2 \\ 0 & c-a & c'-a & cc'-a^2 \\ 0 & d-a & d'-a & dd'-a^2 \end{vmatrix} = (a-b)\begin{vmatrix} 1 & 1 & b+a \\ d-a & d'-a & dd'-a^2 \\ c-a & c'-a & cc'-a^2 \end{vmatrix} = \triangle.$$

Multiply the first row by a and add it to the next two rows.

$$\triangle=(a-b)\begin{vmatrix} 1 & 1 & b+a \\ d & d' & dd'+ba \\ c & c' & cc'+ba \end{vmatrix} =$$

$(a-b)\,[d'cc'+d'ba-c'dd'-c'ba-dcc'-dab+cdd'+abc+bdc'+adc'-bcd'-acd']$ which is the same as $[ab(c+d'-c'-d)+cd'(c'+d-a-b)-c'd(d'+c-a-b)]$ $\times(ab)$ or $\triangle=(a-b)\,[ab(c+d')-ab(c'+d)+cd'(c+d)-cd'(a+b)-c'd[d'+c)$ $+c'd(a+b)]$, which can be written

$$\triangle=(a-b)\left[\begin{vmatrix} ab & a+b \\ cd' & c+d' \end{vmatrix} - \begin{vmatrix} ab & a+b \\ c'd & c'+d \end{vmatrix} + \begin{vmatrix} cd' & c+d' \\ c'd & c'+d \end{vmatrix}\right]$$

which by Cor. 1, page 34, of Hanus' *Elements of Determinants*, is

$$\triangle=(a-b)\begin{vmatrix} 1 & ab & a+b \\ 1 & cd' & c+d' \\ 1 & c'd & c'+d \end{vmatrix}.$$

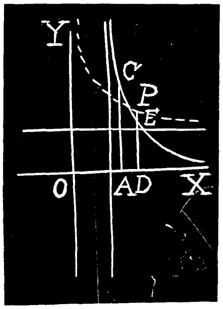
Also solved by J. W. Clawson, and G. B. M. Zerr.

307. Proposed by J. SCHEFFER, Hagerstown, Md.

If $y^x=2$ and $x^y=3$, find x and y.

I. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The solution $(x,\ y)$ of these two equations, is the same as the intersection of the curves (1) and (2).



In the figure, (1) is drawn with full lines, (2) with dotted lines. We see that outside of (0, 0) there is but one point of intersection, P. If we cut the curves by a line $x=c$, it is evident that the value of y from (2) is nearer to the required value than that from (1). If we cut the curve by the line $y=a$, the value of x from (1) is nearer to the required value than the value of x from (2). A rapid approximation can then be found by taking any value of $x>1$ and solving (2) for y. With this value of y substituted in (1), solve for x, etc. If we do this for two values of x, one such that ABC is to the left of P and another such that DEF is to the right of P, we shall approach the point P from both directions and will have an upper and lower limit for the coordinates of P at each step. In this way I find the approximation

$$2.2390<x<2.23939, \quad 1.3627<y<1.36285.$$

Solved similarly by S. Lefseletz.

II. Solution by FRANCIS RUST, C. E., Allegheny, Pa.

The problem proposed by Mr. Scheffer is found in Heis' *Sammlung von Beispielen und Aufgaben, etc.,*" §106, No. 12.

$x^y=b=3$, $y^x=a=2$ makes $y=\dfrac{\log b}{\log x}$, $x=\dfrac{\log a}{\log y}$; *i. e.*, $x\log y=a$, from which is derived the transcendental equation:

$$x\log\,\log b-x\log\,\log x-\log a=0=f(x).$$

Solving same by means of Newton's method:

$$h=-\frac{f(x)}{f'(x)};\quad f'(x)=\log\,\log b-\log\,\log x-\frac{(\log e)^2}{\log x}.$$

$a=2$, $b=3$, $\log e=0.4342945$, results: $x=2.2392498$, $y=1.3628042$.

Also solved by V. M. Spunar, G. B. M. Zerr, J. A. Caparo, A. H. Holmes, and the Proposer.

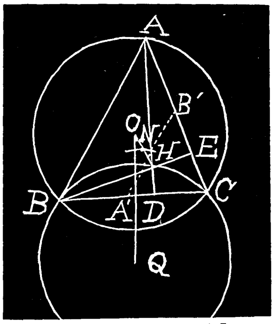
GEOMETRY.

338. Proposed by C. N. SCHMALL, 239 East 7th Street, New York.

Given the base and vertical angle of a triangle, find the locus of the center of its nine-point circle.

I. Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.

(1) Take an angle BAC equal to the given vertical angle and from any point B in one arm of it describe an arc with BC equal to the given base as radius cutting the other arm in C. Draw a circle through A, B, C. Then BC being fixed, the circle ABC is the locus of the vertex A.



(2) Find the center O of this circle, draw OA' the perpendicular bisector of BC, and OB' the perpendicular bisector of CA. Drop AD perpendicular to BC, and BE perpendicular to CA, intersecting AD in H.

The nine-point circle passes through A', D and B', E. Therefore the lines bisecting $A'D$ and $B'E$ at right angles intersect in N, the center of the nine-point circle. But evidently each of these lines bisects OH. Therefore N bisects OH.

(3) Join $A'B'$. The triangles AHB and $OA'B'$ are similar, having their sides mutually parallel. Also $AB=2A'B'$. So $AH=2OA'$, and is therefore constant.

(4) Since AH is constant in length, and as A moves AH moves parallel to itself, the locus of H is a circle of radius equal to OA' and whose center, Q, lies on OA' produced so that $OQ=AH=2OA'$. Draw this circle.

(5) Since N bisects OH, O is a fixed point, and the locus of H is a circle, the locus of N is a circle, whose center bisects OQ, and is therefore the point A', and whose radius is half the radius of the original circle.

Also solved by J. M. Meyer, S. J,, Francis Rust, V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

CALCULUS.

265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find two curves which possess the property that the tangents TP and TQ to the inner one always make equal angles with the tangent TT' to the outer.

II. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Let $y=\phi(x)$ be the equation of the inner curve, and let the coordinates of P and Q, points on this curve, be, respectively, (x_1, y_1), (x_2, y_2). Assuming that the outer curve exists, let its equation be $y=f(x)$, and let $f'(x)=\tan\alpha$.

The two tangent lines from P and Q, respectively, are, if $\phi'(x)=\tan\theta$,

$$y\cos\theta_1 - x\sin\theta_1 + x_1\sin\theta_1 - y_1\cos\theta_1 = 0,$$
$$y\cos\theta_2 - x\sin\theta_2 + x_2\sin\theta_2 - y_2\cos\theta_2 = 0,$$

while the bisector of the angle between these two lines is

$$y(\cos\theta_1 \pm \cos\theta_2) - x(\sin\theta_1 \pm \sin\theta_2)$$
$$+ x_1\sin\theta_1 - y_1\cos\theta_1 \pm (x_2\sin\theta_2 - y_2\cos\theta_2) = 0.$$

Let $(\bar{x}, \bar{y})$ be T, the intersection of the tangents;

$$\therefore \bar{x} = \frac{x_2\phi'(x_2) - x_1\phi'(x_1) + \phi(x_1) - \phi(x_2)}{\phi'(x_2) - \phi'(x_1)},$$

$$\bar{y} = \frac{\phi(x_1)\phi'(x_1) - \phi(x_2)\phi'(x_2) + (x_2 - x_1)\phi'(x_1)\phi'(x_2)}{\phi'(x_2) - \phi'(x_1)}.$$

The normal to the outer curve at T is

$$y\sin\bar{\alpha} + x\cos\bar{\alpha} - \bar{x}\cos\bar{\alpha} - \bar{y}\sin\bar{\alpha} = 0.$$

By the assumption in the given problem, the bisector and normal written above must be identical, hence by comparison of coefficients,

$$\frac{\cos\theta_1 \pm \cos\theta_2}{\sin\bar{\alpha}} = -\frac{\sin\theta_1 \pm \sin\theta_2}{\cos\bar{\alpha}} = \frac{x_1\sin\theta_1 - y_1\cos\theta_1 \pm (x_2\sin\theta_2 - y_2\cos\theta_2)}{-(\bar{x}\cos\bar{\alpha} + \bar{y}\sin\bar{\alpha})}.$$

From the first two members of the last equation, $\tan\bar{\alpha} = \pm\left[\tan\frac{\theta_1+\theta_2}{2}\right]^{\mp 1}$, but since $\tan\bar{\alpha} = f'(\bar{x})$, it follows that $\tan(\theta_1+\theta_2) \equiv \dfrac{\phi'(x_1) + \phi'(x_2)}{1 - \phi'(x_1)\phi'(x_2)}$ should be a function of $\bar{x}$. The condition that one is a function of the other is

$$\frac{\partial}{\partial x_1}\tan(\theta_1+\theta_2)\frac{\partial\bar{x}}{\partial x_2} - \frac{\partial}{\partial x_2}\tan(\theta_1+\theta_2)\frac{\partial\bar{x}}{\partial x_1} = 0.$$

This leads to a relation between x_1 and x_2, and if $\bar{y}$ is to be a function of $\bar{x}$, a similar argument leads to another relation between x_1 and x_2, not necessarily the same as the former. As the original problem assumed no relation whatever between x_1 and x_2, it appears that the problem is indeterminate.

III. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The question is indefinite. Where the inner curve is a conic section, the problem may be solved as follows.

(a) Inner curve an ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0.$$

Let (x_1, y_1) be the point T. Polar of (x_1, y_1) with respect to the conic, is

$$\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1.$$

The curve

$$k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-l\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=0,$$

is a conic tangent to the given ellipse at points of intersection with the polar of T (Charles Smith, *Conic Sections*, p. 203). If this passes through T, it will be the equation of the tangents from T to the conic.

Putting $x=x_1$, $y=y_1$, we find that

$$k=l\left(\frac{x_2^2}{a^2}+\frac{y_1^2}{b^2}-1\right).$$

The equation of the tangents from T is, therefore

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}-1\right)^2=0.$$

The slopes of the lines are determined by the terms of the second degree, which are

$$(b^2-y_1^2)x^2+2x_1y_1xy+(a^2-x_1^2)y^2.$$

The lines bisecting the angles between these lines will be parallel to the lines whose equation is

$$\frac{x^2-y^2}{b^2-y_1^2-a^2+x_1^2}=\frac{xy}{x_1y_1},$$

(Charles Smith, *Conic Sections*, p. 34).

The slope y/x of these lines is determined from

$$\left(\frac{y}{x}\right)^2+\frac{b^2-y_1^2-a^2+x_1^2}{x_1y_1}\left(\frac{y}{x}\right)-1=0.$$

Therefore, the differential equation of the curve of which these lines are tangents is

$$\left(\frac{dy}{dx}\right)^2+\frac{b^2-y^2-a^2+x^2}{xy}\left(\frac{dy}{dx}\right)-1=0.$$

The solution of this equation is the conic section

$$\frac{x^2}{m^2}+\frac{y^2}{n^2}=1,$$

or, in parameter form, $x=m\cos\phi$, $y=n\cos\phi$. From these we get

$$\frac{dy}{dx}=-\frac{n}{m}\cot\phi.$$

Substituting these values of x, y, and dy/dx in the differential equation, we get $m^2-n^2=a^2-b^2$.

Therefore, the required outer curve is any one of a system of ellipses or hyperbolas confocal with the inner curve.

(b) If the inner curve is an hyperbola, we have the same result.

(c) If the inner curve is the parabola, $y^2=4p(x+p)$.

Proceeding in the same way, the equation of the tangents is

$$(y^2-4px-4p^2)(y'^2-4px'-4p^2)-[yy'-2p(x+x')-4p^2]=0.$$

The differential equation is

$$\left(\frac{dy}{dx}\right)^2+\frac{2x}{y}\left(\frac{dy}{dx}\right)-1=0.$$

The solution is, $y^2=4c(x+c)$.

This is a system of parabolas confocal with the given one.

Also solved by the Proposer.

MECHANICS.

222. Proposed by W. J. GREENSTREET, Stroud, England.

Find the maximum angle of inclination to the line of greatest slope of a uniform rod resting on a rough inclined plane and capable of turning freely round a point on it.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

By the maximum angle is evidently meant the angle of limiting equi-

librium. Let R=the reaction of the rod concentrated at G, its center of gravity; μ=coefficient of friction; W=weight of rod; β=inclination of plane to horizon; θ=required angle.

The reaction R at G is perpendicular to the plane, and is $R=W\cos\beta$; while $W\sin\beta$ is the pull down the plane. Now as friction acts opposite to the direction of motion, we get $W\sin\beta\sin\theta=\mu R$.

$\therefore W\sin\beta\sin\theta=\mu W\cos\beta$, or $\sin\theta=\mu\cot\beta$; $\theta=\sin^{-1}(\mu\cot\beta)$.

223. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A sphere, radius $r=\frac{1}{3}$ inches, density $\delta=11.38$, falls from a height $h=$ 500 feet, into a lake depth $l=40$ feet. Find time of falling to surface of lake, time of falling from surface of lake to bottom, and total time of falling. Also the velocity at the bottom.

Solution by the PROPOSER.

Let $r=\frac{1}{3}$ inch$=\frac{1}{36}$ feet=radius of ball; $g=32.2$=gravity; $e_2=g$=weight of unit volume of water; δ=density of air=.001293; $e_1=g\delta$=weight of unit volume of air; R_1v^2=resistance of air; R_2v^2=resistance of water; A=greatest sectional area of sphere$=\pi r^2$; k=a constant=0.51 for the sphere; m= mass of sphere$=\frac{1}{16}g$; v_1=velocity at surface of lake; v_2=velocity at bottom of lake. Then the equations of motion are

$$m\frac{dv}{dt}=mg-Rv^2\ldots(1),\qquad mv\frac{dv}{dx}=mg-Rv^2\ldots(2).$$

From (1), $t=m\int\frac{dv}{mg-Rv^2}$. For the time, T, from rest to the surface of the lake, $R=R_1$, and the limits of v are 0 and v_1.

$$\therefore T=\tfrac{1}{2}\sqrt{\frac{m}{gR_1}}\log\left[\frac{\sqrt{(mg)}+\sqrt{(R_1)}v_1}{\sqrt{(mg)}-\sqrt{(R_1)}v_1}\right]\ldots(3).$$

For the time, T_1, from the surface to the bottom of the lake, $R=R_2$, and the limits of v are v_1 and v_2.

$$\therefore T_2=\tfrac{1}{2}\sqrt{\frac{m}{gR_2}}\log\left[\frac{[\sqrt{(mg)}+\sqrt{(R_2)}v_2][\sqrt{(mg)}-\sqrt{(R_2)}v_1]}{[\sqrt{(mg)}-\sqrt{(R_2)}v_2][\sqrt{(mg)}+\sqrt{(R_2)}v_1]}\right]\ldots(4).$$

From (2), $x=m\int\frac{vdv}{mg-Rv^2}$.

Between the limits v and V we get for x,

$$x=\frac{m}{2R}\log\left[\frac{mg-RV^2}{mg-Rv^2}\right].$$

$$\therefore v^2=\frac{mg}{R}(1-e^{-(2Rx/m)})+V^2e^{-(2Rx/m)}.$$

For the velocity at the surface of the lake, $R=R_1$, $V=0$, $x=h$, $v=v_1$.

$$\therefore v_1^2=\frac{mg}{R_1}(1-e^{-(2h/m)})\ldots(5).$$

For velocity at bottom of lake, $x=l$, $V=v_1$, $R=R_2$, $v=v_2$.

$$\therefore v_2^2=\frac{mg}{R_2}(1-e^{-(2l/m)})+v_1^2e^{-(2l/m)})\ldots(6).$$

Following Rankin: $R_1v^2=\dfrac{kAe_1v^2}{2g}$, $R_2v^2=\dfrac{kAe_2v^2}{2g}$.

$\therefore R_1=\dfrac{0.51\,\pi\,r^2g\,\delta}{2g}=.0000008$; $R_2=\dfrac{0.51\,\pi\,r^2g}{2g}=.000618$.

Since $h=500$, $l=40$, we get: from (5), $v_1=162.447$ feet; from (6), $v_2=10.0564$ feet; from (3), $T=5.814$ seconds; from (4), $T_1=1.773$ seconds.

$T+T_1=7.587$ seconds=total time. These results would be slightly changed for different values of R_1, R_2.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Note on Problem 152.

On referring to the memorandum book from which the problem was taken, I find that p is prime, for which case the theorem would appear to be true. PROPOSER.

155. Proposed by R. D. CARMICHAEL, Anniston, Ala.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

Solution by the PROPOSER.

Some values, found by inspection, are given by Zerr in the MONTHLY for December, 1908. A complete solution may be effected by aid of the following lemma (see *Annals of Mathematics*, Vol. 8, No. 4, p. 15).

If x is a positive integer >1, x^t-1 has a prime factor not dividing x^u-1 $(u<t)$, except in the cases $t=2$, $x=2^v-1$, $v\geqq 2$; $t=6$, $x=2$. Such prime

factors of x^t-1 *are of the form* $st+1$, *and evidently if* t *is odd and* >1 *they are of the form* $2st+1$. Such a prime is called a characteristic factor.

Now, either p or q is even. Suppose that q is even. Then we have

$$2^n=p^m-1.$$

Excluding for the moment the cases of exception in the lemma, we must have $m=1$, since otherwise p^m-1 would have no factor of the form $sm+1$. This gives the solution,

$$(1) \qquad m=1;\ q=2;\ p=2^n+1;$$

where n has any value making 2^n+1 prime. The exceptional case in the present instance is $p=2^v-1$, a prime, and $m=2$. This gives

$$2^n=(2^v-1)^2-1=2^{2v}-2^{v+1}=2^{v+1}(2^{v-1}-1).$$

Hence, $v=2$. Therefore, we have the solution

$$(2) \qquad m=2;\ q=2;\ n=3;\ p=3.$$

Next, suppose that p is even. Then we have

$$2^m=q^n+1.$$

Hence 2 is the characteristic prime factor of $q^{2n}-1=(q^n+1)(q^n-1)$, if it has such characteristic factor. Such factor must be of the form $2sn+1\neq 2$. Hence, we now have no solution except for an exceptional case of the lemma. The only such case applying to the form $q^{2n}-1$ is that for $n=1$, $q=2^v-1$, a prime. This gives $m=v$. Hence, we have the following solution,

$$(3) \qquad n=1;\ q=2^m-1,\text{ a prime};\ p=2;$$

m=any number making 2^m-1 a prime. Hence, all the solutions of the problem are included in (1), (2), and (3).

Remarks by E. B. ESCOTT, Ann Arbor, Mich.

Gerono, *Nouvelles Annales*, 1870, p. 469; 1871, p. 204, has given a demonstration for the case where one of the roots is prime. Catalan says that he wasted a year trying to find a proof. The question was proposed in *L'Intermédiaire des Mathématiciens* as question 487, and the special case $x^2-1=y^3$ as question 664. Father Pepin proved that the last case is impossible except for $x=3$, $y=2$ (*L'Intermédiaire des Mathématiciens*, 1896, pp. 284-5) by using a theorem regarding the impossibility of the equation

$$u^3+x^3=2y^3,$$

which is proven in Legendre, *Théorie des Nombres*, p. 347, and in Euler's *Algebra*, Chap. XV, Part 2. He remarks that neither Euler nor Legendre have justified the use they make of the complex numbers $p+q\sqrt{-3}$. See also, *Crelle*, Vol. 27 (1844), p. 192.

PROBLEMS FOR SOLUTION.

ALGEBRA.

311. **Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.**

Find, by Cardan's Method, the real root (4) of $x^3-6x^2+10x=8$.

312. **Proposed by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.**

Two roots of the cubic $x^3-px^2+qx-c=0$ are equal. Find their value in terms of p, q, and c.

GEOMETRY.

342. **Proposed by G. I. HOPKINS, M. A., Instructor in Mathematics and Astronomy, Manchester, N. H.**

Given, circle DEF inscribed in triangle ABC and circumscribing the triangle DEF, D, E, F being the points of contact; AH is drawn through center, N, meeting chord DF in H. Through H is drawn BK meeting AC in K. Prove triangle ABK isosceles.

CALCULUS.

269. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Prove that $\int_0^1 (x^a+x^{-a})\log\left(\dfrac{1+x}{1-x}\right)\dfrac{dx}{x}=\dfrac{\pi}{a}\tan(\tfrac{1}{2}\pi a)$.

270. **Proposed by S. A. COREY, Hiteman, Iowa.**

Prove that $\sum_{x=0}^{x=\infty}\dfrac{1}{(a^2+x^2)^n}=\dfrac{\pi}{2a^{2n-1}}\cdot\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdots\dfrac{(2n-3)}{(2n-2)}+\dfrac{1}{2a^{2n}}$, n being a positive integer >1.

MECHANICS.

225. **Proposed by W. A. BALDWIN, Senior in Drury College, Springfield, Mo.**

Find, *by means of polar coordinates*, the moment of inertia about the origin of the area between the parabola $ay=2(a^2-x^2)$, the circle $x^2+y^2=a^2$, and the axis of Y.

226. **Proposed by W. J. GREENSTREET, M. A., Stroud, England.**

A frustum of a cone, vertical angle a, is cut off by two spheres whose centers are the vertex. The radius of one sphere is n times that of the other, and the density of the cone varies as the distance of the vertex. Find the ratio into which the centroid of the frustum divides the axis.

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ON THE THEORY OF FUNCTIONS OF A TRIPLE VARIABLE.*

By R. D. CARMICHAEL, Anniston, Alabama.

§1. THE ALGEBRA DEFINED.

Since single real numbers may be represented by the points of a linear continuum we may speak of their algebra as that of one-dimensional extent. Similarly the algebra of double numbers (or number pairs or complex numbers) finds its geometrical image in the points of a surface, or a two-dimensional extent. But no attempt has ever succeeded in defining a system of triple numbers whose algebra is entirely analogous to that of one-dimensional and two-dimensional extent. In fact it has been shown† that no triple algebra exists at the same time conserving all the formal laws of the fundamental operations as developed in our ordinary algebra and also maintaining the important theorem: *If a product of two factors equals zero one at least of the factors is zero.* This theorem plays a fundamental rôle in algebraic analysis and no complete surrender of it can be permitted in constructing a new algebra.

Nevertheless, in attempting to devise a system of units such that numbers built up from them may be imaged in three-dimensional extent, it is just this law which I have agreed to surrender. The problem is then to find a system of units such that the additional complication incident to the surrender of this law shall be reduced to a minimum. Several systems of units with their laws were assumed arbitrarily, but their algebra in every case presented almost insurmountable difficulties. Finally by an analysis of commutative operations in space—translations and rotations—the system of units here employed was defined, first geometrically and then algebraically. We are now concerned with the latter alone.

We begin by defining a system of units by the following equations:

$$\text{I.}\quad r.r=r,\ \frac{r}{r}=r,\ ri=ir=i,\ i^2+r=0,\ s.s=s,\ \frac{s}{s}=s,\ r.s=s.r=i.s=s.i=0.$$

*Read before the American Mathematical Society, October 31, 1908.

†See Stolz and Gmeiner's *Theoretische Arithmetik.*

A number in this system is represented in the form

$$ar+bi+cs,$$

where a, b, c are ordinary reals. It may be looked on as defining a point in space of which the Cartesian coordinates (in a rectangular system) are a, b, c. A more fruitful view of its geometric significance, however, is the following: A plane of reference (chosen at will, but for convenience spoken of as the horizontal plane) has in it rectangular axes intersecting at O. Then the point denoted by $ar+bi+cs$ is c units in perpendicular distance from the plane and has as its projection on the plane the point which in ordinary complex quantities is denoted by $a+bi$. Then every number in this system corresponds uniquely to a point in space; and, conversely, to every point in space there corresponds uniquely a number of this system. The projection on the plane of reference of the general point $ar+bi+cs$ will be denoted not by $ar+bi$ but by

$$ar+bi+0s.$$

We reserve $ar+bi$ for another meaning. It may conveniently be taken to represent a line through the point $ar+bi+0s$ and perpendicular to the pl-ne of reference. It thus includes all points $ar+bi+zs$ where z is a real variable.

Operational and other definitions will follow closely the analogy of the ordinary complex numbers. We define thus the necessary and sufficient condition of equality:

If $\quad ar+bi+cs=\alpha r+\beta i+\gamma s,$
then $\quad a=\alpha,\ b=\beta,\ c=\gamma.$

Addition and subtraction are defined by the equations:

II. Addition: $\quad (ar+bi+cs)+(\alpha r+\beta i+\gamma s)=(a+\alpha)r+(b+\beta)i+(c+\gamma)s.$
III. Subtraction: $\quad (ar+bi+cs)-(\alpha r+\beta i+\gamma s)=(a-\alpha)r+(b-\beta)i+(c-\gamma)s.$

The geometric interpretation is clear. Moreover, these operations obey the usual laws.

If we actually perform the operation of multiplication of two triple numbers, observing the laws of combining the units defined in I, we obtain our multiplication formula, thus:

IV. Multiplication: $\quad (ar+bi+cs)(\alpha r+\beta i+\gamma s)$
$$=(a\alpha-b\beta)r+(a\beta+b\alpha)i+c\gamma s.$$

Adopting the usual definition of division as the inverse of multiplication, we have readily:

V. Division: $\dfrac{ar+bi+cs}{\alpha r+\beta i+\gamma s}=\dfrac{a\alpha+b\beta}{\alpha^2+\beta^2}r+\dfrac{-a\beta+\alpha b}{\alpha^2+\beta^2}i+\dfrac{c}{\gamma}s.$

That the formal laws of ordinary algebra hold for multiplication and division is evident from the form of the results in IV and V.

So far we have not brought our new numbers into relation with ordinary real numbers. The definitions of addition and multiplication enable us to do this. By the formula for addition we have

$$m(ar+bi+cs)=mar+mbi+mcs.$$

The multiplication formula gives the same result for the product,

$$(ar+bi+cs)(mr+0i+ms)=mar+mbi+mcs.$$

Hence we shall define

VI. $$m=mr+0i+ms,$$

and all real numbers are to be combined with triple numbers in accordance with this definition.

We have now to develop the zero-product theorem. Set

$$(ar+bi+cs)(\alpha r+\beta i+\gamma s)=(a\alpha-b\beta)r+(a\beta+b\alpha)i+c\gamma s=0.$$

This requires that

$$a\alpha-b\beta=0,\quad a\beta+b\alpha=0,\quad c\gamma=0.$$

There is no difficulty in verifying that these relations can be satisfied only when one of the factors is of the form $0r+0i+0s$ or when one of them is of the form $0r+0i+cs$ and the other is of the form $\alpha r+\beta i+0s$. We have then the following theorem:

THEOREM. *If the product of two factors is zero, then one of them is zero; or, one of them is of the form $0r+0i+cs$ while the other is of the form $\alpha r+\beta i+0s$.*

There are thus in this system three quantities playing in some respects the rôle of a zero.

It is now easy to see that division is unique and determinate *except in three cases,** namely,

$$\text{VII.} \qquad \frac{0r+0i+0s}{0r+0i+0s}, \quad \frac{0r+0i+cs}{0r+0i+\gamma s}, \quad \frac{ar+bi+0s}{\alpha r+\beta i+0s}.$$

The first is the indeterminate 0/0 of our ordinary complex algebra. The other two have an easy geometric interpretation. For the second, the distance of the point from the plane is fixed, but its projection on the plane is indeterminate. For the third, the projection on the plane of reference is fixed, but the distance from the plane is indeterminate.

If one wishes to interpret for ordinary complex algebra any results obtained by the use of this triple algebra he has merely to omit the terms containing s and to set $r=1$. The formulae and results then become those of ordinary complex algebra.

Our new algebra is now completely defined and related to our former algebra. It is now apparent that *the only added complication or difficulty which can possibly arise will be due to the existence of three indeterminates in division instead of one as in the former algebra.* That this is by no means an insurmountable difficulty becomes apparent as our work progresses.

§2. Functional Dependence.

We shall employ the notation

$$R \equiv xr+yi+zs,$$
$$S \equiv ur+vi+ws,$$

and shall say that S is a function of R when S may be obtained by performing any set of operations upon R considered as a single whole. This of course is precisely analogous to the ordinary conception of a function of a complex variable. Moreover we shall further assume that the functions with which we deal have a diffierential coefficient.

Then if $S=f(R)$, where the form of f is not specified, we have

$$ur+vi+ws=S=f(R)=f(xr+yi+zs).$$

Differentiating with respect to x, y, z, we obtain

$$(1) \qquad \frac{\partial S}{\partial x}=\frac{dS}{dR}\frac{\partial R}{\partial x}=rf'(R)=r\frac{dS}{dR},$$

*Of course this presupposes the introduction of infinity. This is not expressly defined here because it introduces nothing new.

$$(2) \qquad \frac{\partial S}{\partial y}=\frac{dS}{dR}\frac{\partial R}{\partial y}=if'(R)=i\frac{dS}{dR},$$

$$(3) \qquad \frac{\partial S}{\partial z}=\frac{dS}{dR}\frac{\partial R}{\partial z}=sf'(R)=s\frac{dS}{dR}.$$

These equations do not determine any relations existing between w on the one hand and either u or v or both of them on the other. This was to be expected from the definitions of the fundamental operations. In view of these definitions it is evident that

$$w=f(z).$$

But there is a necessary relation between u and v, and it may be developed from the first two differential equations above. We have

$$\frac{\partial S}{\partial x}=r\frac{\partial S}{\partial R}, \quad \frac{\partial S}{\partial y}=i\frac{\partial S}{\partial R}.$$

Multiplying the first by r and the second by i and remembering that $i^2=-r$, we obtain

$$r\frac{\partial S}{\partial x}=-i\frac{\partial S}{\partial y}.$$

Now the relation between u and v is independent of w, and hence in order to find that relation we may ignore w altogether. Geometrically, it amounts to this: For the present we are concerned with the relations between the coordinates of the projections of the variable point on the plane of reference. These are entirely independent of the point's distance from the plane; that is, of w. Therefore from the last equation we may write:

$$r\frac{\partial}{\partial x}(ur+vi)=-i\frac{\partial}{\partial y}(ur+vi).$$

$$\therefore \frac{\partial u}{\partial x}r+\frac{\partial v}{\partial x}i=-\frac{\partial u}{\partial y}i+\frac{\partial v}{\partial y}e. \qquad \therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

These are necessary relations between w and v regarded as functions of x and y. By a method entirely analogous to that employed in the ordinary theory of complex variables they may be shown to be sufficient conditions. Then we have the theorem:

THEOREM I. *In order that we may have*

$$ur+vi+ws=S=f(R)=f(xr+yi+zs)$$

it is necessary and sufficient that the following relations exist:

$$w=f(z), \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

Adding equations (1) and (3) we have

$$\frac{dS}{dR}r+\frac{dS}{dR}s=\frac{\partial S}{\partial x}+\frac{\partial u}{\partial z}.$$

Employing equation VI of the preceding section, this becomes

$$(4) \qquad \frac{dS}{dR}=\frac{\partial S}{\partial x}+\frac{\partial S}{\partial z}.$$

(It is evident that VI holds not only when m is real but also when m is a triple number.) The second member of (4) is evidently independent of the direction in which the point $R+\triangle R$ approaches R; hence

THEOREM II. *The value of the ratio* $\frac{dS}{dR}$ *is independent of the direction in space in which* $R+\triangle R$ *approaches* R.

When in

$$ur+vi+ws=S=f(R)=f(xr+yi+zs)$$

the form of f is given, it clearly suffices theoretically to determine u, v, w. We shall show conversely that if any one of the three quantities u, v, w is given, this alone is sufficient to determine f and the other two of the functions u, v, w.

First, suppose w is given. It must be a function of z alone such that $w=f(z)$, and hence the form of f is determined. It follows as a consequence that u and v have perfectly definite forms.

On the other hand we have only to notice that the relation expressed between u and v in theorem I is exactly that which in the theory of complex variables suffices to determine one of them when the other is given. With both of them determinate, it is evident that the form of f is fixed and therefore w is perfectly definite. Hence,

THEOREM III. *If any one of the functions u, v, w (as u) is given in terms of x, y, z, or any of them, this alone suffices to determine the other two functions (as v and w) and the form of f.*

DUALITY IN THE FORMULAS OF SPHERICAL TRIGONOMETRY.

By W. A. GRANVILLE, Yale University.

It was Moebius* who first called attention to the fact that if we use the supplements of the angles of a spherical triangle instead of the angles themselves, the formulas of Spherical Trigonometry arrange themselves in pairs, either one of a pair being the dual of the other. While this very important notion is now generally employed in advanced treatises dealing with Geometry on the Sphere, I have been unable to discover any writer who has carried out this fruitful idea of duality to its logical conclusion, and that is to apply it to the solution of spherical triangles in a first course in Spherical Trigonometry.

The purpose of this paper is to call the attention of teachers to the many practical advantages attending the use of this property of duality in the formulas of Spherical Trigonometry in teaching the elements of Spherical Trigonometry.

Moebius proved his results for any spherical triangle. For the sake of brevity, however, I shall follow the ordinary practice in a first course and consider such spherical triangles only whose parts are less than 180°.

Given any relation involving one or more of the sides a, b, c, and the angles A, B, C, of any spherical triangle. Now the polar triangle (whose sides are denoted by a', b', c', and angles by A', B', C') is also a spherical triangle and the given relation must hold true for it also; that is, the given relation applies to the polar triangle if accents are placed on the letters representing the sides and angles. Thus, the First Law of Cosines, as usually given, is

$$\cos a=\cos b \cos c+\sin b \sin c \cos A, \tag{1}$$

or, for the polar triangle,

$$\cos a'=\cos b' \cos c'+\sin b' \sin c' \sin A'. \tag{2}$$

But $a'=180^\circ-A$, $b'=180^\circ-B$, $c'=180^\circ-C$, $A'=180^\circ-a$.

Substituting these in (2), we get

$$\begin{aligned}\cos(180^\circ-A)=&\cos(180^\circ-B)\cos(180^\circ-C)\\&+\sin(180^\circ-B)\sin(180^\circ-C)\cos(180^\circ-a),\end{aligned}$$

or (the Second Law of Cosines),

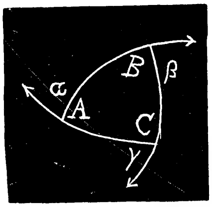
*Moebius: *Ueber eine neue Behandlungsweise der analytischen Sphaerik*, 1846. *Entwicklung der Grundformeln der Trigonometrie in grosstmoeglicher Algemeinheit*, 1860. See *Gesammelte Werke II*.

(3) $$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

which involves the sides and angles of the original triangle. Hence,

THEOREM. *In any relation between the parts of a spherical triangle each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.*

Let the supplements of the angles of the triangle be denoted by α, β, γ; *i, e.* α, β, γ are the exterior angles of the triangle. Then



$$\alpha = 180^\circ - A,\ \beta = 180^\circ - B,\ \gamma = 180^\circ - C,$$
$$\text{or, } A = 180^\circ - \alpha,\ B = 180^\circ - \beta,\ C = 180^\circ - \gamma.$$

When we apply the above theorem to a relation involving the sides and the supplements of the angles of a triangle, we, in fact,

replace a by α $(=180^\circ - A)$,
replace b by β $(=180^\circ - B)$,
replace c by γ $(=180^\circ - C)$,
replace α $(=180^\circ - A)$ by $180^\circ - (180^\circ - a) = a$,
replace β $(=180^\circ - B)$ by $180^\circ - (180^\circ - b) = b$,
replace γ $(=180^\circ - C)$ by $180^\circ - (180^\circ - c) = c$,

or, what amounts to the same thing, *we interchange the Greek and Roman letters.* For instance, substitute

$$A = 180^\circ - \alpha,\ B = 180^\circ - \beta,\ C = 180^\circ - \gamma \text{ in (1).}$$

This gives us the First Law of Cosines in the new form

(4) $$\cos a = \cos b \cos c - \sin b \sin c \cos \alpha.$$

Similarly, we may get

(5) $$\cos b = \cos c \cos a - \sin c \sin a \cos \beta,$$
(6) $$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma.$$

If we now make the substitutions indicated above in (4), (5), (6); that is, interchange the Greek and Roman letters, we get the following new form of the Second Law of Cosines:

(7) $$\cos \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a,$$
(8) $$\cos \beta = \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b,$$
(9) $$\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c;$$

that is, we have derived three new relations between the sides and the sup-

plements of the angles of the triangle. Summing up the results of our discussion, we may then state the following

PRINCIPLE OF DUALITY ON THE SPHERE. *If the sides of a spherical triangle be denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters* α, β, γ, *then from any given formula involving any of these six parts, we may write down a dual formula by simply interchanging the corresponding Greek and Roman letters.*

For the sake of comparison the formulas that are commonly used in the solution of oblique spherical triangles are given in the first column below, while the corresponding new forms are written in the second column.

First Law of Cosines.

$\cos a=\cos b\cos c+\sin b\sin c\cos A$	$\cos a=\cos b\cos c-\sin b\sin c\cos\alpha$
$\cos b=\cos c\cos a+\sin c\sin a\cos B$	$\cos b=\cos c\cos a-\sin c\sin a\cos\beta$
$\cos c=\cos a\cos b+\sin a\sin b\cos C$	$\cos c=\cos a\cos b-\sin a\sin b\cos\gamma$

Second Law of Cosines.

$\cos A=-\cos B\cos C+\sin B\sin C\cos a$	$\cos\alpha=\cos\beta\cos\gamma-\sin\beta\sin\gamma\cos a$
$\cos B=-\cos C\cos A+\sin C\sin A\cos b$	$\cos\beta=\cos\gamma\cos\alpha-\sin\gamma\sin\alpha\cos b$
$\cos C=-\cos A\cos B+\sin A\sin B\cos c$	$\cos\gamma=\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos c$

The formulas for the functions of half the supplements of the angles of a spherical triangle in terms of its sides may be derived in the same manner as the formulas for the functions of half the angles of a triangle in terms of the sides are usually derived. This gives

$\sin\frac{1}{2}A=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$	$\sin\frac{1}{2}\alpha=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}}$
$\cos\frac{1}{2}A=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}}$	$\cos\frac{1}{2}\alpha=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$
$\tan\frac{1}{2}A=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$	$\tan\frac{1}{2}\alpha=\sqrt{\dfrac{\sin s\sin(s-c)}{\sin(s-b)\sin(s-c)}}$

By permuting the letters in cyclical order, we get the remaining two sets of formulas of six each.

The usual method is now to derive the formulas for the functions of the half sides in terms of the angles (shown below in the first column) in like manner. By the use of the Principle of Duality, however, we get the corresponding formulas at once (as shown in the second column) by simply interchanging the Greek and Roman letters in the second column above.

$$\sin\tfrac{1}{2}a=\sqrt{\frac{-\cos A\cos(S-A)}{\cos B\cos C}} \qquad \sin\tfrac{1}{2}\alpha=\sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin\beta\sin\gamma}}$$

$$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}} \qquad \cos\tfrac{1}{2}\alpha=\sqrt{\frac{\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\beta\sin\gamma}}$$

$$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}} \qquad \tan\tfrac{1}{2}\alpha=\sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin(\sigma-\beta)\sin(\sigma-\gamma)}}$$

where $s=\frac{1}{2}(a+b+c)$, $S=\frac{1}{2}(A+B+C)$, $\sigma=\frac{1}{2}(\alpha+\beta+\gamma)$.

Instead of the above two sets of formulas it is sometimes more convenient to use the following two sets when solving triangles.

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}} \qquad \tan\tfrac{1}{2}d=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\tan\tfrac{1}{2}A=\frac{\tan r}{\sin(s-a)} \qquad \tan\tfrac{1}{2}\alpha=\frac{\sin(s-a)}{\tan\tfrac{1}{2}d}$$

etc., where d is the diameter of the inscribed circle, and

$$\tan R=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}$$

$$\tan\tfrac{1}{2}a=\tan R\cos(S-A)$$

$$\tan\tfrac{1}{2}\delta=\sqrt{\frac{\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\sigma}}$$

$$\tan\tfrac{1}{2}\alpha=\frac{\sin(\sigma-\alpha)}{\tan\tfrac{1}{2}\delta}$$

etc., where δ is the supplement of the diameter of the circumscribed circle.

Napier's Analogies in the old and new forms are shown below.

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C \qquad \tan\tfrac{1}{2}(\alpha+\beta)=-\frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C \qquad \tan\tfrac{1}{2}(\alpha-\beta)=-\frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma$$

$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\tfrac{1}{2}(A-B)}{\cos\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c \qquad \tan\tfrac{1}{2}(a+b)=-\frac{\cos\tfrac{1}{2}(\alpha-\beta)}{\cos\tfrac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\tfrac{1}{2}(A-B)}{\sin\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c \qquad \tan\tfrac{1}{2}(a-b)=-\frac{\sin\tfrac{1}{2}(\alpha-\beta)}{\sin\tfrac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c$$

In the case of the Law of Sines the usual form is,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

If we use the supplements of the angles this becomes

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}.$$

If we apply the Principle of Duality to the last form there results

$$\frac{\sin \alpha}{\sin a}=\frac{\sin \beta}{\sin b}=\frac{\sin \gamma}{\sin c};$$

hence, the *Law of Sines goes over into itself.*

In what follows we have in the first column the usual ten formulas used in the solution of right triangles, in terms of α, β, γ. When we apply the Principle of Duality to these we get the formulas in the second column, and these turn out to be formulas for the solution of quadrantal triangles.

	Right Triangle.	*Quadrantal Triangle*
(1)	$\cos c=\cos a \cos b$	$\cos \gamma=\cos \alpha \cos \beta$
(2)	$\sin a=\sin c \sin \alpha$	$\sin \alpha=\sin \gamma \sin a$
(3)	$\sin b=\sin c \sin \beta$	$\sin \beta=\sin \gamma \sin b$
(4)	$\cos \alpha=-\cos a \sin \beta$	$\cos a=-\cos \alpha \sin b$
(5)	$\cos \beta=-\cos b \sin \alpha$	$\cos b=-\cos \beta \sin a$
(6)	$\cos \alpha=-\tan b \cot c$	$\cos a=-\tan \beta \cot \gamma$
(7)	$\cos \beta=-\tan a \cot c$	$\cos b=-\tan \alpha \cot \gamma$
(8)	$\sin b=-\tan a \cot \alpha$	$\sin \beta=-\tan \alpha \cot a$
(9)	$\sin a=-\tan b \cot \beta$	$\sin \alpha=-\tan \beta \cot b$
(10)	$\cos c=\cot \alpha \cot \beta$	$\cos \gamma=\cot a \cot b$

Von Staudt called the expression

$$D=\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c-2\cos a \cos b \cos c}$$

the *sine of the spherical triangle,* and

$$\triangle=\sqrt{1-\cos^2 \alpha-\cos^2 \beta-\cos^2 \gamma-2\cos \alpha \cos \beta \cos \gamma}$$

the *sine of the polar triangle,* because of the relation

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}=\frac{D}{\triangle}.$$

Other authors have called D the *first staudtian* and $\triangle$ the *second staudtian.* We see that either one goes over into the other by the application of the Principle of Duality. This is also shown by the relations

$$D=\sin b \sin c \sin \alpha=\sin c \sin a \sin \beta=\sin a \sin b \sin \gamma,$$
$$\triangle=\sin \alpha \sin \beta \sin a=\sin \beta \sin \gamma \sin b=\sin \gamma \sin \alpha \sin c.$$

Let V denote the volume of the tetrahedron whose vertices are at A, B, C, the vertices of the spherical triangle and O, the center of the sphere. Then it may be shown that

$$6V=r^3 D,$$

where r is the radius of the sphere. If we apply the Principle of Duality to this expression we get

$$6V'=r^3 \triangle,$$

where V' is the volume of the tetrahedron corresponding to the polar triangle A', B', C'.

From a theoretical standpoint the great advantages derived from using the Principle of Duality instead of the usual methods are so apparent that there is scarcely any room for argument. Nearly one-half of the work usually required in deriving the standard formulas is done away with and the resulting formulas are more easily memorized. But, the question now naturally arises, are these new formulas well adapted to numerical calculations? By actual experience in solving a large number of spherical triangles, I have found that there is little or no difference in the amount of labor involved. In the case of given angles we must of course first find their supplements before applying the new formulas, and in the case of required angles we must take the supplements of the angles found from the tables. On the other hand I have found that on account of the duality of the new formulas the detail of the work is simplified and there is less liability of making mistakes. Below is found a solution of a spherical triangle, the new formulas being used.

Example. Given $A=70^\circ$, $B=131^\circ\ 10'$, $C=94^\circ\ 50'$; find a, b, c.

$\alpha=180^\circ-A=110^\circ$
$\beta=180^\circ-B=48^\circ\ 50'$
$\gamma=180^\circ-C=85^\circ\ 10'$
$2\sigma=244^\circ$
$\sigma=122^\circ$
$\sigma-\alpha=12^\circ$
$\sigma-\beta=73^\circ\ 10'$
$\sigma-\gamma=36^\circ\ 50'$

To find $\log\tan\frac{1}{2}\delta$

$\log\sin(\sigma-\alpha)=9.3179$
$\log\sin(\sigma-\beta)=9.9810$
$\log\sin(\sigma-\gamma)=9.7778$
$\text{colog}\sin\sigma=0.0716$
$2|9.1483$
$\log\tan\frac{1}{2}\delta=9.5742$

$\log\sin(\sigma-\alpha)=9.3179$	$\log\sin(\sigma-\beta)=9.9810$	$\log\sin(\sigma-\gamma)=9.7778$
$\log\tan\frac{1}{2}\delta=9.5742$	$\log\tan\frac{1}{2}\delta=9.5742$	$\log\tan\frac{1}{2}\delta=9.5742$
$\log\tan\frac{1}{2}a=9.7437$	$\log\tan\frac{1}{2}b=0.4068$	$\log\tan\frac{1}{2}c=0.2036$
$\frac{1}{2}a=39^\circ$	$\frac{1}{2}b=68^\circ\ 36'$	$\frac{1}{2}c=57^\circ\ 58'$
$a=58^\circ$	$b=137^\circ\ 12'$	$c=115^\circ\ 56'$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

308. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the roots of $x^2+px+q=0$ may not lie between -1 and $+1$.

I. Solution by J. A. CAPARO, University of Notre Dame, Notre Dame, Ind.

In the most general case let one of the roots be $+a$ and the other $-b$, then:

$$(x-a)(x+b)=0 \text{ or } x^2+x(b-a)-ab=0.$$

Comparing with $x^2+px+q=0$, $p=b-a$, $q=-ab$.

The conditions that the roots shall not lie between $+1$ and -1 are:

$$+a>1\ldots(1);\quad +b>1\ldots(2).$$

Multiplying, $ab>1$, but $ab=-q$, therefore $-q>1$ or $q<-1$. Also from (1), (2), $a-1>0$, $b+1>2$.

Multiplying, $ab-b+a-1>0$, or $-ab+(b-a)+1<0$, or $+q+p<-1$. The required conditions then are: $q<-1$ and $q+p<-1$.

II. **Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.**

Assuming that the equation has real roots, *i. e.*, that $p^2>4q$, find Sturm's functions. They are,

$$\left.\begin{array}{l} f(x)=x^2+px+q \\ f'(x)=2x+p \\ f_2(x)=p^2-4q \end{array}\right\}$$

For $+1$, we have, $f(x)=1+p+q$, $f_1(x)=2+p$, $f_2(x)=p^2-4q$.

For -1, we have, $f(x)=1-p+q$, $f_1(x)=-2+p$, $f_2(x)=p^2-4q$.

If there is no root between $+1$ and -1, there will be the same number of variations of signs in each of these. p^2-4q is always positive. We see that the number of variations will be the same if $f(1)$ and $f(-1)$, $f'(1)$ and $f'(-1)$ have the same sign, *i. e.*, $(1+q)^2-p^2>0$ and $p^2-4>0$, *i. e.*,

$$(1+q)^2>p^2>4.$$

They will also have the same number of variations if $f'(1)$ and $f'(-1)$ have opposite signs and $f(1)$ and $f(-1)$ are both negative, *i. e.*, $p^2<4$, $1+q\pm p<0$, or $p\pm q>1$.

Also solved by G. B. M. Zerr, V. M. Spunar, and G. W. Hartwell.

309. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve, $bx^2+cy^2+az^2=ba^2+cb^2+ac^2$,
$cx^2+ay^2+bz^2=ab^2+bc^2+ca^2$,
$xyz=abc$.

Solution by the PROPOSER.

We see by inspection that $x=a$, $y=b$, $z=c$ is one set of solutions.

Then, put $x^2=a^2+u$, $y^2=b^2+v$, $z^2=c^2+w$. The first two equations become $bu+cv+aw=0$, $cu+av+bw=0$. Solving these, we have

$$\frac{u}{a^2-bc}=\frac{v}{b^2-ca}=\frac{w}{c^2-ab}.$$

Put these equal to s, and substitute

$$u=(a^2-bc)s,$$
$$v=(b^2-ca)s,$$
$$w=(c^2-ab)s,$$

in values of x^2, y^2, and z^2:

$$x^2=a^2+(a^2-bc)s,$$
$$y^2=b^2+(b^2-ca)s,$$
$$z^2=c^2+(c^2-ab)s.$$

Substituting these values in the third of the given equations,

$$[s(a^2-bc)+a^2][s(b^2-ca)+b^2][s(c^2-ab)+c^2]=a^2b^2c^2.$$

Two of the roots of this equation are evidently $s_1=0$, $s_2=-1$. The third root can be found by subtracting the sum of these two from the sum of the roots of the equation, which is

$$\frac{a^2}{a^2-bc}+\frac{b^2}{b^2-ca}+\frac{c^2}{c^2-ab}.$$

This gives $s_3=\dfrac{a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2}{(a^2-bc)(b^2-ca)(c^2-ab)}$.

This gives the following solutions:

$$\begin{array}{l|l|l} x=\pm a & \pm\sqrt{bc} & \pm\sqrt{[a^2+(a^2-bc)s_3]} \\ y=\pm b & \pm\sqrt{ca} & \pm\sqrt{[b^2+(b^2-ca)s_3]} \\ z=\pm c & \pm\sqrt{ab} & \pm\sqrt{[c^2+(c^2-ab)s_3]} \end{array}$$

where s_3 has the value fust found.

The signs are to be taken so that xyz is positive.

Also solved by G. W, Hartwell, G. B. M. Zerr, V. M. Spunar, J. Scheffer, J. W. Clawson, and A, H. Holmes.

PROBLEMS FOR SOLUTION.

ALGEBRA.

313. **Proposed by W. J. GREENTTREET, M. A., Stroud, England.**

Find the conditions that the equations $px^2+qx+r=0$ and $\alpha x^2+\beta x+\gamma+y(ax^2+bx+c)=0$ may give equal values for y.

314. **Proposed by R. D. CARMICHAEL, Anniston, Ala.**

Sum to infinity the series $\dfrac{1}{2.3.3.4}+\dfrac{1}{4.5.5.6}+\dfrac{1}{6.7.7.8}+\dfrac{1}{8.9.9.10}+\ldots$

315. **Proposed by PROFESSOR B. F. YANNEY, Mount Union College, Alliance, Ohio.**

Simplify, $1-(2-(3-\ldots-(n-1)-n)\ldots)))$.

316. **Proposed by B. F. FINKEL, Ph. D.**

Prove that $\sum\limits_{r=1}^{r=n}(-1)^{n-1}\dfrac{1}{n}\,{}_nC_r=\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}$ where ${}_nC_r=\dfrac{n(n-1)\ldots.(n-r+1)}{1.2.3\ldots r}$. Dickson's *College Algebra*, ex. 13, p. 92.

GEOMETRY.

343. Proposed by O. J. BROWN, Fairhope, Ala.

From any external point of a triangle, to draw a line so as to divide the triangle into two equal parts.

344. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

A tinsmith has a sheet of copper in the form of a rectangle, sides a and b. He desires to cut this into two pieces which will form a square when placed together. How can he do this?

CALCULUS.

171. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-5\left(\frac{d^2y}{dx^2}\right)\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\tfrac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

show that there is an integrating factor of the form $\left(\frac{d^2y}{dx^2}\right)^n$, and integrate the equation.

172. Proposed by CLARENCE OHLENDORF, Chicago, Ill.

Find $\int \log_e \tan^{-1} x dx$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

160. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that the integer next above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

161. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find a solution of $x^4-5x^2+4\equiv 0 \pmod{p.\overline{2p+1}}$, where both p and $2p+1$ are odd primes.

162. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

If p is an odd prime, find the number of incongruent integers x for which x^4+2ex^2+f is a quadratic residue of p.

MISCELLANEOUS.

181. Proposed by A. H. HOLMES, Brunswick, Me.

In latitude 43° 45′ N., the sun's declination being 16° 30′ N., at what time in the forenoon will the angle on the horizon between the east point and the foot of the meridian passing through the sun's position be equal to its altitude?

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FAMILIES OF CENTRAL ORBITS RELATED TO CIRCULAR TRAJECTORIES.*

By FRANK LOXLEY GRIFFIN, Williams College.

§ 1. CIRCLES AS CENTRAL ORBITS.

1. THE PROBLEM. Given any central force which is everywhere finite and real, and which is a function of the distance alone, $f=f(r)$, it is known that all circles with centers at the center of force are possible orbits. For if a particle were to move in such a circle, of radius a, the centrifugal force v^2/a would have to balance exactly the attraction $f(a)$; and, conversely, if the particle were started at a distance a from the center of force, in a direction perpendicular to the radius vector, and with the velocity defined by $v^2=af(a)$, the circular orbit would result.

But in order that a circle whose center is elsewhere than at the center of force, be described as a central orbit, the force must vary according to a *special* law, which is obtainable when the polar equation of the given orbit has been written. The object of this paper is (1) to derive such a law of force, (2) to ascertain what other circular orbits are admitted by it and what properties have they in common, (3) to learn what other laws admit families of orbits having some of those properties, and (4) to point out the families in question in the case of two simple laws.

2. LAWS ADMITTING CIRCULAR ORBITS. Let the center of force be selected as the pole, and let the polar angle θ be measured from the line through the pole and the center of the circle; then the equation of the orbit is

$$2d\cos\theta=1/u-(a^2-d^2)u, \tag{1}$$

where u is the reciprocal of the radius vector, a is the radius of the circle, and d is the distance between the pole and center. [It is necessary to take

*Read before the American Mathematical Society, September 10, 1908.

$d<a$, since a closed central orbit not containing the center of force is impossible.*]

The required central force must vary along the orbit according to the law: $f=h^2u^2(u+d^2u/d\theta^2)$, where h is the constant of areas. The differentiation of (1) with respect to θ, followed by the elimination of θ from the two equations gives

(2) $$4d^2=[1/u^2-(a^2-d^2)]^2u^2+[1/u^2+(a^2-d^2)]^2(du/d\theta)^2,$$

whence is obtained by adding $4(a^2-d^2)$ to both members:

(3) $$u^2+(du/d\theta)^2=4a^2u^4[(a^2-d^2)u^2+1]^{-2}.$$

By differentiating again, or by noting that the curvature, given by $u^3(u+d^2u/d\theta^2)[u^2+(du/d\theta)^2]^{-\frac{3}{2}}$, equals $1/a$ at all points of the circle, the necessary force is easily obtained as a function of u alone:

(4) $$f=\frac{m^2u^5}{(n^2u^2+1)^3}=\frac{m^2r}{(n^2+r^2)^3},$$

where $m^2=8a^2h^2$ and $n^2=a^2-d^2$. Some obvious properties of this force are: (a) it vanishes both "at infinity" and at the center of force; (b) it is everywhere real; and (c) it is everywhere finite and continuous. If the values of a, d, and h had been different in the given orbit, the values of m and n would usually be different; but the properties mentioned would persist. The type of law where the force is given by (4) for particular values of m and n will be called Type I hereafter.

§ 2. Other Circular Orbits for Laws of Type I.

3. A Family of Circles. Consider any particular law of Type I. Because of its properties, just enumerated, this law admits the infinitude of circular orbits whose centers are at the pole. But, leaving aside this somewhat trivial family, the question arises as to the possibility of further circular orbits besides the original one. Evidently in all such orbits which may be admitted by the given law, a^2h^2 and a^2-d^2 must be invariant, for if either m or n changes value, the law changes.

The constancy of the second expression restricts the set of circles admissible as orbits; that of the first imposes a restriction upon the initial conditions. The totality of possible circles is represented by the equation

(5) $$2D\cos(\theta-K)=1/u-n^2u,$$

*Cf. E. J. Routh, *Dynamics of a Particle*, pp. 292-3.

obtained from (2) by replacing the original d and a by D and A the corresponding constants for any other circle, imposing the restriction $A^2-D^2=n^2$, and introducing K as the longitude of the center of the circle. There is a double infinitude of these circles, obtained by varying D and K.

Moreover, as will now be shown, there exist real initial conditions for which any one of these circles is described as a central orbit.*

4. CIRCULAR ORBITS THROUGH A POINT. *Theorem I. Through any point* (u_1, θ_1) *there passes in each direction one and only one circle of the family* (5). For, if the given direction makes with the radius vector to that point an angle ψ_1, then, since $\tan\psi=r\,d\theta/dr=-u\,d\theta/du$, equation (3) gives for any circle of (5) on replacing a by A and a^2-d^2 by n^2:

$$u_1^2\csc^2\psi_1=4A^2u_1^4[n^4u_1^2+1]^{-2}. \tag{6}$$

This equation determines a single real positive value of A, which is never less than n, since $\csc^2\psi_1\geq 1$ and $n^2u_1^2+1\geq 2nu_1$:

$$A^2=\csc^2\psi_1.(n^2u_1^2+1)^2/4u_1^2\geq n^2. \tag{7}$$

Hence $D^2=A^2-n^2$ gives a real value for D. The substitution of the latter value, with the coordinates of (u_1, θ_1) in (5) determines K uniquely (except for additive multiples of 2π) and real.

Theorem II. There is a unique velocity with which a particle projected at any point in any direction will describe a circle. For, by the preceding theorem, A is determined uniquely; and hence $HA=m$ determines H uniquely. And since $H=v_1r_1\sin\psi_1$ in any central orbit, there is but one possible value for v_1. Now to describe the required circle with the determined value of H, the particle would have to be attracted by a force whose law is precisely (4); hence, conversely, projection in the given direction with the velocity v_1 will result, under the law (4), in a circular orbit.

5. PROPERTIES OF THE FAMILY OF ORBITS. From the constancy of AH it follows that the constant of areas required in any circular orbit decreases as the radius of the circles increases. From the constancy of A^2-D^2 it follows that the radius of the circle must always increase with the distance of its center from the pole. There is no upper limit for A or D, but there is a lower limit (not zero) for A: viz., the smallest possible circular orbit is that one whose center lies at the center of force. But the two invariants have more exact interpretations, which are both simple and interesting.

*In a paper "On the Law of Gravitation in the Binary Systems," [*American Journal of Mathematics.* vol. 31, pp. 62-85], the writer discussed the family of conics by a law of which Type I is a limiting case. The present family of circles, however, is not included among those conics; for each law there considered admitted only one circular orbit, and the center was at the pole. Certain questions raised and answered below would not have arisen in connection with the family of conics.

Property I. The curvature at either apse is proportional to the constant of areas required for the orbit. For the curvature is $1/A$ which equals H/m.

Property II. The product of the two apsidal distances is constant. For these distances are $A-D$ and $A+D$ whose product equals n^2.

Property III. All circular orbits whose apsidal radii vectores are collinear pass through two fixed points on the perpendicular drawn to the line of centers at the center of force. For the half-chord intercepted on this perpendicular, being a mean proportional between $A-D$ and $A+D$, is constant and equal to n

Another (and more useful) statement of this property is that *the vectorial angle from either apse to the radius vector of length n is constant and equal to $\pi/2$.*

§ 3. Other Laws Admitting Properties I—III.

6. Admitting Property I. Two of these three properties were found also in the family of conics mentioned above; and it is natural to inquire whether there are still other laws of force whose trajectories will possess such properties. Attention will be confined to those cases where the force is a continuous function of the distance.

Let the force be denoted by $u^2P(u)$, and let $F(u)=2\int P(u)\,du$. [Since the continuity of $P(u)$ follows from the hypothesis that f is a continuous function of r, $P(u)$ is integrable; thus $F(u)$ exists and is continuous.]

A first observation is that any such force, whether attractive or repulsive, admits families of orbits having pericenters at arbitrary distances; and, if everywhere attractive, admits also families having apocenters at arbitrary distances. For a particle may be so projected at any distance $1/\beta$ that the initial direction is perpendicular to the initial radius vector, so that $du/d\theta=0$ for $u=\beta$; and the initial velocity may be such as to give h any desired value. Moreover the standard differential equation of central orbits,

$$f=h^2u^2(u+d^2u/d\theta^2)=u^2P(u), \tag{8}$$

shows that $d^2u/d\theta^2 \gtrless 0$ at $u=\beta$ according as $h^2 \gtrless P(\beta)/\beta$. Hence whatever the sign of $P(\beta)$, h may be chosen so large that $d^2u/d\theta^2<0$, making u a maximum at the initial point. And, if $P(\beta)>0$ [force attractive], it is also possible to choose h so small that u is a minimum at $u=\beta$.

Secondly, since the expression for the curvature [see No. 2] becomes at an apse $u+d^2u/d\theta^2$, or simply $P(u)/h^2$, Property I would require: $h=\lambda P(\beta)/h^2$, or

$$h^3=\lambda . P(\beta), \tag{9}$$

where λ is a constant factor of proportionality. Thus for *any* value of λ, h is defined as a function of β so as to insure Property I. It remains to be shown simply that λ may have such values that h, defined by (9), will meet the condition above for the existence of a pericenter or apocenter at $u=\beta$. Now to have $h^2>P(\beta)/\beta$ is to have $[\lambda P(\beta)]^2>[P(\beta)/\beta]^3$, or $\lambda^2>P(\beta)/\beta^3$. But for any interval of β, $0<A\leqq\beta\leqq B$, $P(\beta)$ being continuous has an upper bound M. Hence, if λ^2 be chosen greater than M/A^3, the condition is satisfied. And if $P(\beta)>0$ always, the reverse inequality may be treated in like manner.

Therefore, *every central force which is a continuous function of the distance admits families of orbits having pericenters and possessing Property I; and, if everywhere attractive, families of orbits with apocenters, having that property.*

Forces admitting orbits with both a pericenter and an apocenter are somewhat special as will appear in the next paragraph. For the orbits to have Property I at their pericenters only, any continuous force whose trajectories have two apses is admissible, the corresponding apocenters being determined as follows: One integration of (8) gives

$$h^2(u^2+(du/d\theta)^2)=c+F(u), \tag{10}$$

where c is an arbitrary constant. If β and α denote, respectively, the pericentral and apocentral values of u, then

$$h^2(\beta^2-\alpha^2)=F(\beta)-F(\alpha). \tag{11}$$

Now if β be allowed to vary, and h be determined by (9), then (11) will determine α in those cases where the orbit has two apses. If, however, Property I is to hold for both apses, there is a further restriction upon the force, given by $\lambda P(\beta)=\mu P(\alpha)$, μ being the factor of proportionality for apocenters. A case of this sort is treated later [in No. 9].

7. ADMITTING PROPERTY II. If an orbit has two apsidal distances, $1/\alpha$ and $1/\beta$, Property II requires simply that $\alpha\beta=k^2$, a constant. As remarked, however, there is a second category of forces none of whose trajectories have two apsidal distances; an example is the case where the force varies inversely as the cube of the distance. Still other laws admit trajectories consisting of two branches separated by a region of imaginary motion, —one branch having only a pericenter and extending to infinity, the other having only an apocenter and reaching to the center of force. In such a case which branch is actually described in the motion of the particle depends upon the initial distance of the particle; and the path really has but one apse. It is easy to find conditions upon the force, which are necessary and sufficient to place it in any of the three categories.

Theorem I. In order that a force, f, shall admit orbits extending from one apse to another, it is necessary and sufficient that f/u^3 be a decreasing function of u throughout some interval of u, say $[a, b]$. For if there is an apocenter at $u=\alpha$ and a pericenter at $u=\beta$ $[\beta>\alpha]$, the necessary constant of areas is given by (11); and moreover it is necessary that $h^2d^2u/d\theta^2$, or $-h^2u+P(u)$, be positive for $u=\alpha$ and negative for $u=\beta$; thus

$$P(\alpha)/\alpha>h^2>P(\beta)/\beta. \tag{12}$$

If $P(u)/u$ is nowhere a decreasing function of u, it is clear that the inequality (12) is impossible; thus the hypothesis is necessary.

It is also sufficient; for let $\alpha=a$ and $\beta=b$. Then the use of the generalized theorem of mean value in (11) gives the constant of areas for which $du/d\theta$ vanishes at $u=\alpha$ and at $u=\beta$:

$$h^2=\frac{F(\beta)-F(\alpha)}{\beta^2-\alpha^2}=\frac{P(u_1)}{u_1}, \tag{13}$$

where $\alpha<u_1<\beta$, the theorem being valid here, since $F(u)$ and u^2 together with their derivatives are continuous functions in $[\alpha, \beta]$, and the latter dedominator does not vanish in the interval. Now by hypothesis,

$$\frac{P(\alpha)}{\alpha}>\frac{P(u_1)}{u_1}>\frac{P(\beta)}{\beta}; \tag{14}$$

so that (12) is satisfied, and there is an apocenter where $u=\alpha$ and a pericenter where $u=\beta$. Whether the orbit extends from $u=\alpha$ to $u=\beta$ or whether there are still other apses between these values of u (for the same values of h) does not affect the theorem. For consider the next apsidal value above $u=\alpha$, say $u=\alpha'$. Then since $h^2=P(u_2)/u_2$ where $\alpha<u_2<\alpha'$; and since this shows that $h^2>P(\alpha')/\alpha'$, the apse at $u=\alpha'$ is a pericenter. A similar argument shows that the apse nearest to $u=\beta$, say $u=\beta'<\beta$, would have to be an apocenter. In any event the same branch has both a pericenter and an apocenter.

Corollary 1. If f/u^3 decreases everywhere, then any two distances can be the apsidal distances in an orbit. For the constant of areas required for the vanishing of $du/d\theta$ would be such as to give a pericenter at the lesser distance and an apocenter at the greater.

Theorem II. In order that a force, f, admit trajectories consisting of two branches, one having an apocenter only, and one a pericenter only, it is sufficient that f/u^3 be everywhere an increasing function of u, and necessary that f/u^3 be an increasing function in some interval $[a, b]$. For the existence of a pericenter at $u=\beta$ and an apocenter at $u=\alpha$ $[\alpha>\beta]$ requires the

inequality (12), which can be satisfied only if f/u^3 is an increasing function in some interval between β and α. Further, since (14) is satisfied if f/u^3 is everywhere an increasing function, the apses at $u=\alpha$ and $u=\beta$ must be, respectively, an apocenter and a pericenter; and there are no further apses by Theorem I, the necessary condition not being fulfilled.

Corollary 2. If f/u^3 is constant [the case of the inverse cube of the distance], *only one apse is possible in a trajectory.* For the conditions necessary for two apses either in the same branch or in different branches are not satisfied.

Corollary 3. There is no apsidal distance between $1/\alpha$ and $1/\beta$ in the orbits of Theorem I. For between the apocentral value $u=\beta'$ and the pericentral value $u=\alpha'$, f/u^3 would have to be somewhere an increasing function.

Theorem III. Any law admitting trajectories with two apsidal distances admits families having Property II.

Case I. When there is an interval $[a, b]$ throughout which f/u^3 decreases, any two apsidal values α and β $[a \leqq \alpha < \beta \leqq b]$ determine a value of h for which the orbit lies in the region $u=a$ to $u=b$ and has two apses. Let $ab=k^2$, and choose arbitrarily $k \geqq \alpha \geqq a$, and $\beta=k^2/\alpha$. Then the orbit lies between $u=\alpha$ and $u=\beta$, and has Property II.

Case II. When f/u^3 always increases with u, select k and α arbitrarily $[\alpha>k]$, and let $\beta=k^2/\alpha$. Then the vanishing of $du/d\theta$ for $u=\alpha$, β, determines a value of h, $h^2=P(u_1)/u_1$, such that $P(\alpha)/\alpha<h^2<P(\beta)/\beta$. Thus the trajectory has two branches of the kind considered in Theorem II. By varying α a family is obtained having Property II.

8. Admitting Property III. Not all forces admit orbits in which the angle from the radius vector of a given length to the nearest pericenter is $\pi/2$. For example, the laws represented by $f=kr^n$ where $n>1$ have their apsidal angles all less than $\pi/2$.* But for any force, f, such that $u.f$ is an increasing function of u, all apsidal angles are greater than $\pi/2$; and by Corollary 1, if also f/u^3 is everywhere a decreasing function, any pair of apsidal values α and β are admissible.

Let c be an arbitrarily chosen constant; and let θ denote the angle from the radius vector where $u=c$ to the nearest pericentral line, where $u=\beta>c$. Then

$$(15) \qquad \theta=\int_c^\beta \frac{du}{\sqrt{\{\beta^2-u^2+[F(u)-F(\beta)]/h^2\}}}.$$

Now for sufficiently great values of h, this integral differs arbitrarily little from

*This and the following statement are established by a test contained in an unpublished paper by the writer, presented to the American Mathematical Society, December 30, 1908.

$$(16) \qquad \int_{\beta}^{c} du/\sqrt{(\beta^2-u^2)}, \text{ or } \pi/2-\arcsin(c/\beta);$$

hence there are values of h large enough to make $\theta<\pi/2$. But if α be chosen sufficiently near c [$\alpha<c$], $\Theta-\theta$ may be made arbitrarily small [Θ denoting the apsidal angle]; so that, since $\Theta>\pi/2$, h can be so chosen that $\theta>\pi/2$. Between this value of h and the "sufficiently great values," there must exist a value for which $\theta=\pi/2$.

By varying β a family of orbits is obtained having Property III. For any value of β the necessary value of h is given by (15) on placing $\theta=\pi/2$; and α is given by (11) when β and h are known.

There is likewise a family having Property III with respect to the apocenters; for with a given α, h can be selected so small that θ shall be arbitrarily small; or β selected so near c that θ shall be greater than $\pi/2$. Hence, for a certain value of h, $\theta=\pi/2$.

The determination of h in each case is unique; for in (15) if h be assigned two different values h_1 and h_2, $h_2>h_1$, the integrand involving h_2 is smaller throughout than that involving h_1, so that only for one of these values can $\theta=\pi/2$.

9. Admitting Properties I and II. The preceding sections show that no one of the three properties serves to characterize any particular law of force. But it is quite otherwise if laws be sought admitting both Properties I and II.

For if the constant of areas be proportional to the curvature at each apse; then $h^3=\lambda P(\beta)=\mu P(\alpha)$, where λ and μ are constants. And if the family is to have $\alpha\beta=k^2$, λ and μ must be equal, since as α and β each approach the value k, h^3 approaches $\lambda P(k)$ or $\mu P(k)$. Therefore, also $P(k^2/\beta)=P(\beta)$; and from (11) follows

$$(17) \qquad F(\beta)-F(k^2/\beta)=(\beta^2-k^4/\beta^2)[\lambda.P(\beta)]^{2/3},$$

which must hold as β varies through the range of values taken in the family. Differentiation of (17) with respect to β gives on replacing $P(k^2/\beta)$ by $P(\beta)$:

$$(18) \qquad \begin{aligned}(1+k^2/\beta^2)P(\beta)=(\beta+k^4/\beta^3)&[\lambda P(\beta)]^{2/3}\\ &+(\lambda/3)(\beta^2-k^4/\beta^2)[\lambda P(\beta)]^{-1/3}P'(\beta),\end{aligned}$$

or substituting kx for β and $\phi(x)$ for $\lambda.P(kx)$:

$$(19) \qquad \frac{d\phi}{dx}+3\frac{x^4+1}{x^5-x}\phi=\frac{3}{\lambda k}\cdot\frac{\phi^{\frac{4}{3}}}{x^2-1},$$

a differential equation which is of Bernoulli's type, and is reducible to a lin-

ear equation by the substitution $\phi = y^{-3}$. A particular integral is $y = (x^2+1)/2\lambda kx$, while the solution of the auxiliary equation is $y = c\sqrt{(x^4-1)}/2\lambda kx$; so that the general solution is

(20) $$\phi^{-1/3} = y = [c\sqrt{(x^4-1)} + x^2 + 1]/2\lambda kx,$$

where c is an arbitrary constant; or

(21) $$P(\beta) = 8k^6\lambda^2\beta^3[c\sqrt{(\beta^4-k^4)} + \beta^2 + k^2]^{-3}.$$

Hence the most general law of force, $f = u^2P(u)$, would be

(22) $$f = 8k^6\lambda^2 \cdot \frac{u^5}{[c\sqrt{(u^4-k^4)} + u^2 + k^2]^3}.$$

But, whether c is real or imaginary $[c \neq 0]$, (22) gives imaginary values for f either when $u > k$ or else when $u < k$; and such a force would not admit real orbits in both parts of the plane. Thus the only admissible value of c is zero; and (22) then reduces to the same form as (4).

Hence *laws of Type I are the most general which admit families of orbits possessing Properties I and II;* and these two properties serve, therefore, to characterize laws admitting circular orbits whose centers are elsewhere than at the center of force.

§ 4. The Families for Particular Laws.

10. For Newton's Law. Every orbit described under a force operating according to Newton's law is a conic having a facus at the center of force. Expressed in terms of the constants of the conic, the force is given by

(23) $$f = h^2u^2/a(1-e^2),$$

so that, for all the orbits, $h^2/a(1-e^2)$ must be constant. If also h varies as the curvature at any apse, $1/a(1-e^2)$ or a/b^2, it follows that $a(1-e^2)$, and hence also h, must be constant in the family. Hence, for the Newtonian law, the family having Property I is the *totality of conics* having a certain curvature at a vertex, or what is equivalent, *having latera recta of a certain length.* Evidently there is an infinitude of such families, obtained by changing the arbitrary constant. And further, each of these families possesses Property III also.

Again, to have the product of the apsidal distances constant requires the constancy of $a^2(1-e^2)$, or of the minor axis. Hence for Newton's law,

the family possessing Property II is the set of conics having minor axes of an arbitrary constant length.

11. FOR THE LAW OF THE DIRECT DISTANCE. All orbits described under a force varying directly as the distance, are conics whose centers lie at the center of force. The force, expressed in terms of the constants of the conics, is given by

$$f=h^2/a^2b^2u; \tag{24}$$

so that h^2/a^2b^2 must be constant for all the orbits. If also h is proportional to a/b^2, it is necessary that b^6, and hence also b, remain invariant. Thus, for this law, the family possessing Property I is the set of conics having minor axes of a given length. The constant of areas must vary as the major axis in this family. To have Property I at the pericenters, interchange b and a in the preceding statements. Each of these families has Property III also.

Finally, to have the product of the apsidal distances constant requires the constancy of $a.b$ and also of h. Hence, for the law of the direct distance, the family possessing Property II is the set of conics having the product of the axes constant, or what is equivalent for ellipses, having a given (arbitrary) area.

ON THE IRREDUCIBILITY OF CERTAIN POLYNOMIALS.

By JACOB WESTLUND, Purdue University.

The object of the following note is to determine whether the two polynomials

$$f_1(x)=(x-a_1)(x-a_2)\ \dots\ (x-a_n)-1 \text{ and}$$
$$f_2(x)=(x-a_1)(x-a_2)\ \dots\ (x-a_n)+1,$$

where a_1, a_2, ..., a_n are distinct integers, are reducible or irreducible.

Let us first consider $f_1(x)$. If $f_1(x)$ were reducible we would have

$$f_1(x)=\phi(x)\psi(x),$$

where $\phi(x)$ is irreducible and of a lower degree than n. Then since

$$\phi(a_i)\psi(a_i)=-1,\ i=1,\ 2,\ \dots,\ n,$$

we must have

$$\phi(a_i) = \pm 1 \text{ and } \psi(a_i) = \mp 1.$$

Hence,

$$\phi(a_i) + \psi(a_i) = 0,\ i = 1, 2, \ldots, n;$$

and hence the equation

$$\phi(x) + \psi(x) = 0,$$

whose degree is less than n, has n distinct roots, which is impossible. Hence, $f_1(x)$ is always irreducible.

Let us next consider $f_2(x)$. If $f_2(x)$ were reducible, we would have

$$f_2(x) = \phi(x)\psi(x),$$

where $\phi(x)$ is irreducible and of a lower degree than n. Then reasoning in the same way as in the first case we find that the equation

$$\phi(x) - \psi(x) = 0,$$

which is of a lower degree than n, has n distinct roots. But this is impossible, unless $\phi(x)$ and $\psi(x)$ are identically equal. Hence the only case when $f_2(x)$ is reducible is when it is a perfect square, in which case n of course must be even.

A METHOD OF COMPUTING LOGARITHMS.

By C. E. WHITE, Nashville, Tennessee.

From the expansion $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$ we derive by letting $a = x+h$ and $a = x-h$,

$$f(x) = f(x+h) - hf'(x+h) + \frac{h^2}{2!}f''(x+h) - \frac{h^3}{3!}f'''(x+h) + \ldots$$

$$f(x) = f(x-h) + hf'(x-h) + \frac{h^2}{2!}f''(x-h) + \frac{h^3}{3!}f'''(x-h) + \ldots$$

The above expansions may be used to an advantage in computing log-

arithms if h be determined so that $x+h$ and $x-h$ are each equal to some number whose logarithm is known.

Let us choose h in the first series so that $x+h=1000$ and h in the second series so that $x-h=1000$.

$$f(x+h)=\log 1000,\ f'(x+h)=\tfrac{1}{1000}=.001,\ \frac{f''(x+h)}{2\,!}=.0000005.$$

$$\therefore \log x=\log 1000-(.001h+.0000005h^2+.000000000333h^3+\ldots) \text{ or}$$
$$\log_{10}x=3-M(.001h+.0000005h^2+.000000000333h^3+\ldots)$$

For the second series we get

$$\log_{10}x=3+M(.001h-.0000005h^2+.000000000333h^3+\ldots)$$
$$\log 999=3-.43429\times .001-.43429\times .0000005=2.99956549$$
$$\log 1001=3+.43429\times .001-.43429\times .0000005=3.00043407$$
$$\log 997=3-.43429(.003+.0000045)=2.99869516$$
$$\log 1003=3+.43429(.003-.0000045)=3.00130093$$

The advantage of this method in computing a table of logarithms is that two logarithms can be computed with little more work than is required to compute one.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A very good solution of 309 was received from J. M. Arnold.

210. Proposed by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Simplify, $\log[\sqrt[3]{(137)}\ \sqrt[7]{(56)}\div\sqrt[10]{(187)}\ \sqrt[4]{(75)}]$.

Remark by the PROPOSER.

This problem presents no difficulty whatsoever. It was proposed with a view of ascertaining whether different results would be obtained. Only two solutions have been received. We should like to have a large number of solutions.

311. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Find, by Cardan's Method, the real root (4) of $x^3-6x^2+10x=8$.

Solution by G. I. HOPKINS, Instructor in Mathematics and Astronomy, High School, Manchester, N. H.

Substitute $y+2$ for x. Then $y^3-2y-4=0$.
Substitute $v+z$ for y. Then, from the well known formula,

$$v=\sqrt[3]{[2+\tfrac{10}{9}\sqrt{3}]} \text{ and } z=\sqrt[3]{[2-\tfrac{10}{9}\sqrt{3}]}.$$

$\therefore 2+\frac{10}{9}\sqrt{3}$ must be the cube of a binomial, the first term of which is 1, and the second term contains $\sqrt{3}$.

Assume $(1+a\sqrt{3})^3=2+\frac{10}{9}\sqrt{3}$; whence $1+3a\sqrt{3}+9a^2+3a^3\sqrt{3}=2+\frac{10}{9}\sqrt{3}$, or $3a(1+a^2)\sqrt{3}+9a^2=1+\frac{10}{9}\sqrt{3}$.

$\therefore 9a^2=1$; whence $a=\frac{1}{3}$.

$\therefore 1+\frac{1}{3}\sqrt{3}$ is the cube root of $(2+\frac{10}{9}\sqrt{3})$.

$\therefore v=1+\frac{1}{3}\sqrt{3}$ and $z=1-\frac{1}{3}\sqrt{3}$; $v+z=2=y$; and, therefore, $x=y+2=4$.

Also by quadratics as follows:

$$\begin{aligned}
\text{Multiplying by } x,\ & x^4-6x^3+10x^2-8x=0.\\
& x^4-6x^3+9x^2+x^2-8x=0,\\
& (x^2-3x)^2+x^2-3x-5x=0,\\
& (x^2-3x)^2+(x^2-3x)=5x.\\
\text{Adding } x^2-3x,\ & (x^2-3x)^2+2(x^2-3x)=x^2+2x.\\
\text{Adding } 1,\ & (x^2-3x)^2+2(x^2-3x)+1=x^2+2x+1.
\end{aligned}$$

$\therefore x^2-3x+1=\pm(x+1)$ and $x=4$.

Also solved by J. Scheffer, G. B. M. Zerr, G. W. Hartwell, and a student in Olivet College.

CALCULUS.

268. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Determine $\phi(y)$, independent of u, so that the equation $\int_0^u (u-y)^{(2p-1)/2}\phi(y)\,dy=u^m$ is satisfied, p and m being positive integers and $m>p$. Do you notice properties of special interest for any special cases?

Solution by the PROPOSER.

The problem may be generalized with small increase in the difficulty of solution; thus:

It is required to find $\phi(y)$ a function of y alone from the equation

$$(1)\qquad f(u)=\int_0^u h(u,\,y)u^a\,\phi(y)\,dy,$$

where $h(u, y)$ is a homogeneous function of u, y of degree zero such that it may be expressed as a finite or an infinite series,

$$h(u, y)=\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\beta_3\frac{y^3}{u^3}+\ldots,$$

and where $f(u)$ is a given function which may be written in the form

$$f(u)=c_1 u^{a_1}+c_2 u^{a_2}+c_3 u^{a_3}+\ldots, \quad \alpha \leqq a_1<a_2<a_3<\ldots;$$

α and the a's being positive or negative, entire or fractional, subject to the given relations.

First consider the special case

$$c_i u^{a_i}=\int_0^u\left(\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\ldots\right)u^\alpha\,\phi_i(y)\,dy. \tag{2}$$

Now

$$\int_0^u\left(\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\ldots\right)\delta_i y^{a_i-\alpha-1}dy$$

$$=\delta_i u^\alpha\left(\frac{\beta_0}{a_i-\alpha}y^{a_i-\alpha}+\frac{\beta_1}{a_i-\alpha+1}\,\frac{y^{a_i-\alpha+1}}{u}+\frac{\beta_2}{a_i-\alpha+2}\,\frac{y^{a_i-\alpha+2}}{u^2}+\ldots\right)\Bigg]_0^2$$

$$=\delta_i\, s_i\, u^{a_i},\text{ where} \tag{3}$$

$$s_i=\frac{\beta_0}{a_i-\alpha}+\frac{\beta_1}{a_i-\alpha+1}+\frac{\beta_2}{a_i-\alpha+2}+\frac{\beta_3}{a_i-\alpha+3}+\ldots. \tag{4}$$

Equating the first member of (2) with the second of (3), we have

$$\delta_i=\frac{c_i}{s_i}. \quad \therefore \phi_i(y)=\frac{c_i}{s_i}y^{a_i-\alpha+1}.$$

Evidently, then, a solution of (1) is

$$\phi(y)=\sum_i\phi_i(y);\text{ or }\phi(y)=\sum_i\frac{c_i}{s_i}x^{a_i-\alpha-1},$$

where i runs over all the subscripts of c in the expression for $f(u)$ and s_i is defined by equation (4).

NOTE.—Volterra has studied a more general problem. See *Encyklop. d. Math. Wissensch.*, II, p. 808. The problem in the present form affords the solution of several problems in practical hydrodynamics.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

157. **Proposed by A. H. HOLMES, Brunswick, Maine.**

Find integral values for m and n in $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$.

No solution of this problem has been received.

158. **Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.**

Find positive rational values of a and b in the equation $x^4-2ax^2+x+a^2-b=0$, that will make each of the roots (all different) rational numbers.

Solution by E. B. ESCOTT, Ann Arbor, Mich.

Let two roots be $x=\alpha$ and $x=\beta$. Substituting, we have

$$b=\alpha^4-2a\,\alpha^2+\alpha+a^2=\beta^4-2a\,\beta^2+\beta+a^2.$$

Transposing, and removing factor $\alpha-\beta$ (since $\alpha\neq\beta$), we have $\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3-2a(\alpha+\beta)+1=0$, *i. e.*,

$$a=\tfrac{1}{2}\left(\alpha^2+\beta^2+\frac{1}{\alpha+\beta}\right).$$

Substituting, we get $b=\frac{1}{4}\left(\alpha^2-\beta^2-\dfrac{1}{\alpha+\beta}\right)^2+\alpha$.

Substituting in the original equation, and removing the factors $x-\alpha$ and $x-\beta$, we have

$$x^2+(\alpha+\beta)x+\left(\alpha\beta-\frac{1}{\alpha+\beta}\right)=0.$$

This will have commensurable roots if

$$(\alpha+\beta)^2-4\left(\alpha\beta-\frac{1}{\alpha+\beta}\right)=(\alpha-\beta)^2+\frac{4}{\alpha+\beta}=r^2.$$

It is easily seen that α and β cannot be integral.

We can get as many rational values as we please by assuming any value for $\alpha+\beta$.

Example. Let $\alpha+\beta=\frac{1}{2}$. Then $\beta=\frac{1}{2}-\alpha$. Substituting in the last equation, we get $(2\alpha-\frac{1}{2})^2+8=r^2$. This can be satisfied in an infinite number of ways, *e. g.*,

$$2\alpha-\tfrac{1}{2}+r=4,\quad 2\alpha-\tfrac{1}{2}-r=-2,$$

whence $\alpha=\frac{3}{4}$, $\beta=-\frac{1}{4}$. Then $a=\frac{21}{16}$, $b=\frac{21}{16}$. The other roots are $\frac{5}{4}$ and $-\frac{7}{4}$.

Also solved by G. B. M. Zerr, and V. M. Spunar.

AVERAGE AND PROBABILITY.

192a. Proposed by A. H. HOLMES, Brunswick, Maine.

In a game of baccarat the dealer and each side of the table have two or three cards. The object is to get as near nine as possible, and tens and court cards do not count. If the first two cards dealt do not together amount to five, the player asks for another. If above five he does not. When the two cards amount to exactly five would the chances of the hand be bettered or diminished by drawing a third card, and how much?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let A, B, C, D be the players, and, in order to avoid a multiplicity of solutions, we will assume cards for A, C, D when B has just five. Let A have an ace and a three; C, four and six; D, two and five, as follows:

A	C	D	B
1, 3	4, 6	2, 5	1, 4 or 2, 3 or 5, 10

In either case B betters his hand if he draws 1, 2, 3, 4, 5, 6, or 7; diminishes it if he draws 9 and leaves it the same if he draws 8, 10, or court cards. Since there are but 44 cards left in the pack, we have:

First $\left\{\begin{array}{l} \frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{4}{44}=\frac{10}{22}=\frac{5}{11}=\text{chance of bettering.} \\ \frac{4}{44}=\frac{1}{11}=\text{chance of diminishing.} \\ \frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}=\frac{5}{11}=\text{chance of leaving same.} \end{array}\right.$

Second $\left\{\begin{array}{l} \frac{3}{44}+\frac{2}{44}+\frac{2}{44}+\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{4}{44}=\frac{5}{11}=\text{chance of bettering.} \\ \text{The chance of diminishing or leaving the same} \\ \text{is } \frac{1}{11} \text{ or } \frac{5}{11} \text{ as before.} \end{array}\right.$

Third $\left\{\begin{array}{l} \frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{3}{44}+\frac{2}{44}+\frac{3}{44}+\frac{4}{44}=\frac{21}{44}=\text{chance of bettering.} \\ \text{The chance of diminishing}=\frac{4}{44}=\frac{1}{11}. \\ \frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{4}{44}+\frac{3}{44}=\frac{19}{44}=\text{chance of remaining the same.} \end{array}\right.$

The solution is similar for A, C, D having other cards.

Hence we see that the drawing of a third card when the two cards count just five, is preferable. In the first and second cases, bettering to diminishing=5 : 1. In the third case, bettering to diminishing=21 : 4.

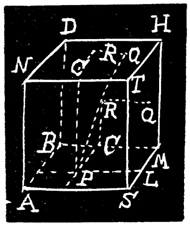
197. Proposed by HENRY HEATON, Belfield, N. D.

Solve No. 188 on the supposition that all lines having the same direction are equally distributed in space, and lines passing through the same point are distributed as the radii of a sphere drawn to points equally distributed.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

We will solve (b) first.

Let AH be the cube, P the point of ingress, Q the point of egress of the hole. Let (x, y) be the coordinates of P. Through P, pass the plane RPC parallel to the face $ABDN$. Through Q draw the line QR parallel to BM and DH, where $BMDH$ is a face of the cube. Let $LM=x$, $LP=y$, $\angle RPC=\theta$, $\angle QRP=\phi$, $a=$ side of cube. When P, Q are in the adjacent faces $ASMB$ and $BMHD$, then $PR=x\sec\theta$, $PQ=x\sec\theta\sec\phi$. The limits of θ are 0 and $\tan^{-1}(a/x)=\theta_1$; of ϕ, 0 and $\tan^{-1}(y\cos\theta/x)=\phi_1$. When P is in $ASMB$ and Q in the opposite face $NTHD$, $PR=a\sec\theta$, $PQ=a\sec\theta\sec\phi$. The limits of θ are 0 and $\tan^{-1}(x/a)=\theta_2$; of ϕ, 0 and $\tan^{-1}(y\cos\theta/a)=\phi_2$; the limits of both x and y are 0 and a.



As P may be regarded as fixed, there are four faces adjacent to $ASMB$ and one opposite. The points of intersection of PQ with the surface of a sphere, center P and radius unity, is the required distribution. An element of surface of this sphere is $\cos\phi\, d\theta\, d\phi$. Hence the average length, L, is

$$L=\frac{\int_0^a\int_0^a\left[4\int_0^{\theta_1}\int_0^{\phi_1}x\sec\theta\sec\phi\cos\phi\, d\theta\, d\phi\right.}{\int_0^a\int_0^a\left[4\int_0^{\theta_1}\int_0^{\phi_1}\cos\phi\, d\theta\, d\phi\right.}\times$$

$$\times\frac{\left.+\int_0^{\theta_2}\int_0^{\phi_2}a\sec\theta\sec\phi\cos\phi\, d\theta\, d\phi\right]dx\,dy}{\left.+\int_0^{\theta_2}\int_0^{\phi_2}\cos\phi\, d\theta\, d\phi\right]dx\,dy}=\frac{N}{D}.$$

$$D=\int_0^a\int_0^a\left[4\int_0^{\theta_1}\frac{y\cos\theta}{\sqrt{(x^2+y^2\cos^2\theta)}}d\theta+\int_0^{\theta_2}\int\frac{y\cos\theta}{\sqrt{(a^2+y^2\cos^2\theta)}}d\theta\right]dx\,dy$$

$$=\int_0^a\int_0^a\left[4\sin^{-1}\left(\frac{ay}{\sqrt{(a^2+x^2)}\sqrt{(x^2+y^2)}}\right.\right.$$

$$\left.\left.+\sin^{-1}\left(\frac{xy}{\sqrt{(a^2+x^2)}\sqrt{(a^2+y^2)}}\right)\right]dx\,dy=\int_0^a\left[4a\sin^{-1}\left(\frac{a^2}{a^2+x^2}\right)\right.$$

$$-4x\log\left[\frac{[a+\sqrt{(a^2+x^2)}]\sqrt{(a^2+x^2)}}{x[a+\sqrt{(2a^2+x^2)}]}\right]+a\sin^{-1}\left(\frac{x}{\sqrt{(2)}\sqrt{(a^2+x^2)}}\right)$$

$$-x\log\left(\frac{\sqrt{(2)}\,[x+\sqrt{(a^2+x^2)}]}{x+\sqrt{(2a^2+x^2)}}\right)\Bigg]dx$$

$$=(a^2/12)\,[10\pi+24-45\sqrt{2}+21\sqrt{3}+6\log 2-24\log(1+\sqrt{3})+27\log(1+\sqrt{2})]$$
$$=2.6653a^2.$$

$$N=4\int_0^a\int_0^a\int_0^a \frac{x}{\sqrt{(x^2+z^2)}}\tan^{-1}\left(\frac{y}{\sqrt{(x^2+z^2)}}\right)dz\,dx\,dy$$

$$+\int_0^a\int_0^z\int_0^a \frac{a}{\sqrt{(a^2+z^2)}}\tan^{-1}\left(\frac{y}{\sqrt{(a^2+z^2)}}\right)dz\,dx\,dy$$

In the first term z is written for $x\tan\theta$, in the second term z is written for $a\tan\theta$.

$$N=4\int_0^a\int_0^a\left[\frac{ax}{\sqrt{(x^2+z^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(x^2+z^2)}}\right)\right.$$

$$\left.-\tfrac{1}{2}x\log\left(\frac{a^2+x^2+z^2}{x^2+z^2}\right)\right]dz\,dx+\int_0^a\int_0^z\left[\frac{a^2}{\sqrt{(a^2+z^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(a^2+z^2)}}\right)\right.$$

$$\left.-\tfrac{1}{2}a\log\left(\frac{2a^2+z^2}{a^2+z^2}\right)\right]dz\,dx=\int_0^a\left[4a\sqrt{(a^2+z^2)}\tan^{-1}\left(\frac{a}{\sqrt{(a^2+z^2)}}\right)\right.$$

$$\left.-4az\tan^{-1}(a/z)+2z^2\log\left(\frac{a^2+z^2}{z\sqrt{(2a^2+z^2)}}\right)\right]dz$$

$$+\int_0^a\left[\frac{a^2z}{\sqrt{(a^2+z^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(a^2+z^2)}}\right)-\tfrac{1}{2}az\log\left(\frac{2a^2+z^2}{a^2+z^2}\right)\right]dz$$

$$=\frac{a^3}{12}\left[28\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+8\log 2-24-7\pi-4\log 3\right]$$

$$+4a^3\int_0^{\frac{1}{4}\pi}\sec^3\psi\tan^{-1}(\cos\psi)\,d\psi. \qquad 4a^3\int_0^{\frac{1}{4}\pi}(\sec^3\psi\tan^{-1}(\cos\psi)\,d\psi$$

$$=4a^3\int_0^{\frac{1}{4}\pi}(\sec^2\psi-\tfrac{1}{3}+\tfrac{1}{5}\cos^2\psi-\tfrac{1}{7}\cos^4\psi+\tfrac{1}{9}\cos^6\psi-\tfrac{1}{11}\cos^8\psi+\tfrac{1}{13}\cos^{10}\psi-\tfrac{1}{15}\cos^{12}\psi$$

$$+\ldots)\,d\psi=4a^3(1.02697-.06868\pi)=3.2448a^3.$$

$$\therefore N=1.5390a^3,\quad L=N/D=.5774a.$$

$$\text{(a)}\quad L=\frac{2a\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos^2\phi\,d\theta\,d\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\cos\phi\,d\theta\,d\phi}=\frac{4a}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos^2\phi\,d\theta\,d\phi=a.$$

MISCELLANEOUS.

174. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

By a linear transformation with integral coefficients modulo 2, reduce $f=\Sigma x_i^2+\Sigma x_ix_j$ $(i, j=1, \ldots, 2m;\ i<j)$ to a canonical form in which the variables are separated into pairs.

Solution by the PROPOSER.

Let $x_1=y_1+s$, $x_2=y_2+s$, where $s=\sum\limits_{i=3}^{2m} x_i$. Then (mod 2),

$$f\equiv y_1y_2+y_1^2+y_2^2+F,\quad F=\Sigma\, x_ix_j\ (i, j=3, \ldots, 2m;\ i<j).$$

In F, set $x_3=y_3+t$, $x_4=y_4+t$, where $t=\sum\limits_{i=5}^{2m} x_i$. Then

$$F\equiv y_3y_4+G,\quad G=\Sigma\, x_i^2+\Sigma\, x_ix_j\ (i, j=5, \ldots, 2m;\ i<j),$$

modulo 2. Since G is of the form f, repetitions of the process evidently lead to the required canonical form.

PROBLEMS FOR SOLUTION.

ALGEBRA.

317. Proposed by FRANCIS RUST, Allegheny, Pa.

Once, in classic days, Silenus lay asleep; a goat skin filled with wine near him. Dionysius passing by, profited, by siezing the skin, and drinking for two-thirds ($\frac{2}{3}$) of that time in which Silenus alone could have emptied said skin. At this point Silenus awoke, and seeing what was happening, snatched away the precious skin, and finished it.

Now, had both started together, and drank simultaneously, they would have consumed the wine skin in two hours less time. And, in this case, Dionysius' share would have been one-half as much as Silenus did secure, by waking and snatching the skin.

In what time would either one of them alone finish the goat-skin?

CALCULUS.

173. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

On one side of a circular pond a feet in radius is a duck. On the diametrically opposite side of the pond is a dog. Both begin to swim at the same time, the duck swimming around the circumference of the pond at the rate of m feet a minute, the dog swimming directly towards the duck at the rate of n feet per minute. How far will the dog swim in overtaking the duck?

174. **Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.**

About the vertices of a regular tetrahedron four spheres are drawn with radii equal to the edge of the tetrahedron. Find the volume common to them all.

175. **Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.**

Explain fully why the circular measure of an angle is used in the calculus.

MECHANICS.

227. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

228. **Proposed by J. E. ROSE, Mount Angel College, Mount Angel, Oregon.**

AB, BC are two uniform rods freely hinged at B, whose weights are W, $4W$, and lengths $2a$, $4a$, respectively. The ends A, C of the rods are attached to small rings which slide on a rough horizontal wire. When the distance between the rings is the greatest for which equilibrium can exist, both of them are on the point of slipping. Find the coefficient of friction.

229. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the area being vertical, the bounding axis major being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

230. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

A particle is projected from a distance $a=2r$ from the earth's center towards the earth with a velocity from infinity. If the earth is an airless homogeneous sphere, radius equal to the present mean radius and gravity as at present, with what velocity and in what time will it reach the center through an opening from surface to center?

BOOKS.

Plane and Spherical Trigonometry and Four-Place Tables of Logarithms. By William Anthony Granville, Ph. D., Sheffield Scientific School, Yale University. 8vo. Cloth. xi+264 pp. +38 pp. of Tables. Price, $1.25. Boston and Chicago: Ginn & Co.

The subject of Trigonometry as presented in this book is in accordance with the latest and most approved methods. Trigonometric Functions are defined as ratios, first for angles less than a right angle and then extended to angles in general. A valuable feature of the book is the fact that the degree of accuracy which may be expected in a result found by the aid of the tables is clearly indicated. In the treatment of oblique spherical triangles the author makes use of the Principle of Duality, whereby nearly half of the work usually required in deriving the standard formulae is saved and the usual six cases are reduced to three. A large number of illustrative problems are worked out under each topic and a large number of well graded exercises are given. The typography and mechanical execution of the book are very beautiful and artistic. B. F. F.

An Introduction to the Study of Integral Equations. By Maxime Bôcher, B. A., Ph. D., Professor of Mathematics in Harvard University. 8vo. Paper cover. 3+72 pages. Price, 2s, 6d. Cambridge: Cambridge University Press.

In this tract, the author has presented the main portions of the Theory of Integral Equations in a readable and accurate form. It has followed roughly the line of historical development, beginning with Abel's Mechanical Problem. It is believed that the careful student is here furnished with a firm foundation upon which he may build further study and investigation, while the more superficial reader will be satisfied with the concise and precise statements of results. B. F. F.

Spezielle Ebene Kurven, von Dr. Heinrich Wieleitner, Gymnasiallehrer am hum. Gymnasium Speyer, mit 189 Figuren im Text. Gr. 8^0. xvi+409 Seiten. Preis, in Leinwand gebunden, M. 12. Leipzig: G. J. Goeschen'sche Verlogshandlung.

This is a very valuable book on special algebraic and trancendental plane curves. Here is to be found a full treatment of the Cissoid, Conchoid, Lemniscate, Cartesian Ovals, Roulettes, Spirals, etc., etc. Many generalizations of these curves are made and their properties are derived. The book is of special value to the student of modern Geometry.

B. F. F.

Correlation of Efficiency in Mathematics and Efficiency in Other Subjects. A Statistical Study. By Professor H. L. Rietz, Ph. D., and Miss Imogene Shade, A. B., November, 1908. 20 pages. Price, 35 cents.

This is the latest number of *The University Studies*, published by the University of Illinois. It is a scientific comparison of the grade of work students do in mathematics, foreign languages and natural sciences. The data, covering a period of nineteen years, were procured from the Registrar of the University of Illinois. The method of investigation may be characterized as the statistical method of Galton and Pearson. The paper presents some new points in the theory of statistics, but the main result of general interest is the discovery that a student who is good in mathematics is also good in foreign languages or in natural sciences, or *vice versa.* The authors call attention to the important educational value of these results. L. J. P.

First Course in Calculus. By E. J. Townsend, Ph. D., (Goettingen) Professor of Mathematics, University of Illinois, and G. A. Goodenough, M. E. (Illinois), Associate Professor of Mechanical Engineering, University of Illinois. Large 8vo. Cloth. xii+466 pages. Price, $2.00. New York: Henry Holt & Co.

Among the excellent works on the Calculus which have recently appeared, the one under consideration must be accorded a very high rank. The presentation of the fundamental principles of the Calculus are rigorously established, the arrangement of subjects is logical, and the material selected eminently practical. The illustrations and applications from Physics are commendable. The book is one well suited for class room purposes and also for self-instruction and supplementary reading. This work is a worthy contribution to the literature of the subject. B. F. F.

The Principles of Mechanics for Students of Physics and Engineering. By Henry Crew, Ph. D., Fayerweather Professor of Physics in Northwestern University. 8vo. Cloth. ix+295 pp. Price, $1.50. New York: Longmans, Green & Co.

The author's efforts, as stated in the preface, have been: (1) To lead the student to clear dynamical views in the shortest possible time, without sacrificing him on the altar of logic, yet pursuing a route which he can afterwards follow with safety; (2) to build the discussion upon a few simple experiments and upon definitions which convey at once the physical meaning of the quantities defined; (3) to follow the example of Föppl in using vector analysis merely to present a clear, simple, and accurate picture of the facts, reserving the Cartesian analysis for purposes of computation; (4) to confine the treatment to that part of mechanics which is common ground for the physicist and the engineer; (5) to reduce the inherent difficulties of the subject to a minimum by treating dynamics in two analogous parts—rotational and translational—such that if either one is given the other may be immediately deduced; and (6) to employ only two systems of units, the absolute C. G. S. and the "British Engineers." In our opinion, the author has executed his purpose most admirably and has given the teacher of mechanics a teachable book.

B. F. F.

A Manual of Practical Physics for Students of Engineering. By Ervin Sidney Ferry, Professor of Physics, Purdue University, and Arthur Taber Jones, Assistant Professor of Physics, Purdue University. Vol. I. Fundamental Measurements and Properties of Matter. Heat. 8vo. Cloth. xi+273 pp. Price, $1.75. New York: Longmans, Green & Co.

This work furnishes the student of pure and applied science with a manual of the theory and manipulation of those measurements which bear most directly upon his work in other departments of study and also upon his professional career. The description and theory of each experiment is very clear and the material well selected. However, the book, in our judgment, would have been improved had the authors inserted a blank form for the data and results of each experiment. It is the purpose of the authors to complete the course in three volumes, the last two of which are in preparation. B. F. F.

Théorie et Applications des Equations du Second Degré a l'usage des Eléves de Seconde et Primiere C et D et de Mathematiques A et B. Par J. Juhel-Rénoy. Paris: Vubert et Nony Editeurs. 251 pages.

This book treats very fully the equation of the second degree, also rational fractions whose numerators and denominators are of the second degree, and inequalities of the second degree. Biquadratic equations reducible to quadratics are discussed in Book III, chapter I and reciprocal equations in chapter II of this book and irrational equations in the last chapter, III. B. F. F.

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PERIODIC DECIMAL FRACTIONS.

By ELIZABETH R. BENNETT, The University of Illinois.

Although decimal fractions present themselves very early in the mathematical course yet they offer many questions to which it is difficult to find an answer in the text-books or even in the mathematical encyclopedias. Some of these questions are of such an elementary type that their answers can be appreciated even by students of little mathematical training. Moreover, periodic decimals offer "an unlimited amount of material for practice, leading to results whose beauty and easy verification have a great charm."*

The main object of the present paper is to collect the principal theorems relating to periodic decimals and to present them in a form which can easily be understood. It is believed that a useful service may thus be rendered to teachers of secondary mathematics, especially since these theorems are so scattered and relate to such an elementary subject. In addition to this, the writer has proved some of the results by more modern methods, especially by using elementary properties of group theory. These methods of proof are the only elements of novelty claimed for the present article. An outline of these methods was published by Professor Miller, *Bulletin of the American Mathematical Society*, Vol. XIV, 1908, page 356.

In the *Nouvelles Annales de Mathêmatiques* for 1842 M. Catalan has summarized the principal theorems then known regarding decimals, basing the greater number of his proofs on Fermat's theorem. Another summary by Laisant and Beaujeux which includes several additional theorems appears in the same magazine for 1868. The proofs given in this article are developed from the standpoint of radix fractions, *i. e.*, fractions developed according to the powers of a base. Since only a few of the theorems stated in these two summaries, or elsewhere, are found in the mathematical encyclopaedias or in the texts on arithmetic, algebra, or number theory, it has seemed desirable to present briefly the more important ones in this article. Although no attempt has been made to exhibit new theorems, in several instances, the proofs have been modernized by the application of elementary group theory principles.

*Weber und Wellstein, *Encyklopaedie der Elementar-Mathematik*, Vol. 1, 1906, p. 258.

Possibly the theorems which are best known and most frequently found in encyclopaedias and texts are the three which will be stated first. Proofs for the first and third of these theorems may be found, for instance, in Chrystal's *Algebra*, second edition, Part I, page 171, while the proof for the second theorem is almost self-evident.

It is assumed in the theorems that will be stated that m/n is a proper fraction in lowest terms.

Theorem I. The necessary and sufficient condition that a decimal terminates is that n must be of the form $2^{\alpha} 5^{\beta}$ where α and β may both be positive integers or either α or β may be zero.

Theorem II. If n contains powers of 2 and 5 as well as other factors, the powers of 2 and 5 may be removed, and after a certain number of places, the fraction will have the same mantissa as some fraction with a denominator prime to 10.

Theorem III. Any rational fraction m/n, n being prime to 10, is periodic.

The general rules which govern the number of places in the decimal period, considering the fraction as having unity for a numerator will now be given, and then it will be shown that the length of this period is independent of the value of the numerator.

Theorem IV. Given any fraction $1/p$, p being an odd prime, finding the period of this fraction consists simply in finding the exponent to which 10 belongs, modulus p.

If 10 is a primitive root of p, the period will be of length $p-1$. If 10 is not a primitive root of p, the exponent will be some divisor of $\phi(p)$, according to Fermat's theorem.

Theorem V. If in the fraction $1/n$, n is the product of different odd primes, the period of $1/n$ is equal to the least common multiple of the periods of the primes into which n can be resolved.

This theorem is proved by the following theorem: If $n=p^{\alpha} q^{\beta} r^{\gamma} \ldots$, where p, q, r, etc., are different primes, and if f, g, h, etc., are the exponents to which a belongs, moduli p^{α}, q^{β}, r^{γ}, ..., then t being the least common multiple of f, g, h, ..., $a^{t} \equiv 1$, modulus n. When $a=10$, $10^{t} \equiv 1$, modulus n, or the required period is of length t.

Theorem VI. If n is of the form p^{α} and $\dfrac{10^{s}-1}{p}$ is prime to p, s being the exponent to which 10 belongs, modulus p, then the length of the resulting period is equal to the length of the original period multiplied by $p^{\alpha-1}$.

When 10 is a primitive root of p and also a primitive root of p^{2}, then 10 belongs to exponent $p-1$, modulus p, and to exponent $\phi(p^{2})$ or $p(p-1)$, modulus p^{2}. From the theory of primitive roots it is known that in order that 10 may be a primitive root of p and also a primitive root of p^{2}, $\dfrac{10^{p-1}-1}{p}$ must be prime to p. Since any primitive root of p^{2} is also a primi-

tive root of p^a, the theorem holds for p^a. If 10 is not a primitive root of p, 10 must belong to some exponent which is a divisor of $p-1$ and it may be shown that in this case the theorem also holds.

Theorem VII. The length of the period of m/n is independent of the value of the numerator.

If n is any given number, there are always $\phi(n)$ different numbers less than n and prime to n. Then if m/n is any proper fraction in lowest terms, there are always $\phi(n)$ different fractions with a denominator n having numerators less than n and prime to n. But the $\phi(n)$ different numbers less than n and prime to n form a group in respect to multiplication, modulus n. Therefore, the numerators of the given fractions also form a group in respect to multiplication, modulus n.

For every fraction m/n, n prime to 10, 10 will be one of the $\phi(n)$ numbers less than n and prime to n, if $10<n$. If $10>n$, some residue of 10, modulus n, will occur. The powers of 10, modulus n, form a cyclic sub-group G_1 of the larger multiplication group G.

All the fractions having numerators belonging to G_1 will have a period of the same length as $1/n$. Assume that $10^x \equiv 1$, modulus n, or that the period of $1/n$ contains x places. Then $1/n = .a_1a_2\ldots a_xa_1a_2\ldots$ Multiplying by 10 simply changes the decimal point each time one place to the right, or the length of the period for each fraction whose numerator belongs to G_1 is of the same length as that of $1/n$. There is only a cyclical interchange of the numbers composing the period.

An operator of G not in G_1 multiplied by an operator of G_1 will give some distinct element of the group not in the cyclic sub-group. Let k be such an operator in G. Then suppose $k/n = .\beta_1\beta_2\beta_3\ldots\beta_1\beta_2\beta_3\ldots$ Multiplying by 10 simply moves the decimal point one place to the right, therefore, the multiples of k/n by powers of 10 will have a period of the same length as k/n. But it has been assumed that 10 belongs to exponent x, modulus 10 or $10^x \equiv 1$, modulus n, therefore, $k/n \times 10^x$ gives a period of the same length as that of the original period of the cyclic sub-group. If the operators of G are not yet exhausted, another operator not already used may be chosen and the above reasoning repeated. It is then clear, since all fractions with numerators belonging to G_1 have the same period as $1/n$ and all others have the same period as fractions whose numerators belong to G_1 that the length of the period is independent of the value of the the numerator. A concrete example will serve to illustrate these statements. Assume $n=21$. Then $\phi(21)=\phi(3)\phi(7)=12$ and the fractions having numerators prime to 21 are as follows: $\frac{1}{21}$, $\frac{2}{21}$, $\frac{4}{21}$, $\frac{5}{21}$, $\frac{8}{21}$, $\frac{10}{21}$, $\frac{11}{21}$, $\frac{13}{21}$, $\frac{16}{21}$, $\frac{17}{21}$, $\frac{19}{21}$, $\frac{20}{21}$. In this case the cyclic sub-group G_1 of the numerators will be 1, 10, 16, 13, 4, 19. The periods of the fractions having these numbers for numerators will be found by a cyclical interchange of the the numbers .047619. Taking 2 as an operator in G another set of six numbers would be given by the fractions $\frac{2}{21}$, $\frac{20}{21}$, $\frac{11}{21}$, $\frac{5}{21}$, $\frac{8}{21}$, and $\frac{17}{21}$. The periods would be a cyclical interchange of the numbers .095238.

The number of different cyclic sets, or different periods of the same length, for each denominator n is equal to $\phi(n)$ divided by the exponent to which 10 belongs, modulus n. This is true since $\phi(n)$ gives the number of fractions having numerators in the multiplication group G and the exponent to which 10 belongs, modulus n, gives the number of these fractions in each cyclic set. For instance, the number 10 belongs to exponent 6, modulus 21, and $\phi(21)$ is 12. Therefore, there are two different cyclic sets for $n=21$, as was seen in the previous example.

The operation of finding the numbers composing the different cyclic sets of the $\phi(n)$ fractions is very much shortened by the fact, that, in general, the different cyclic sets, or periods, occur as complements of each other. Since -1 and $+1$ always occur among the $\phi(n)$ numbers of G, the $\phi(n)$ fractions whose numerators differ only in respect to sign, modulus n, have periods occurring in complementary pairs. Then if a period has been obtained, its complementary period is found by subtracting each digit of the first period from nine. If 10 is a primitive root of n, then -1 is in the cyclic sub-group G_1. The period is in this case of even length and the second half of the period is obtained by subtracting the digits of the first half from nine. The period is of even length since G_1 then contains an operator of order two and the order of the operator must divide the order of G_1. If -1 is not in G_1, then the index of G_1 under G must be even and the periods are complementary.

THE DEFLECTING FORCE OF THE EARTH'S ROTATION AND FOUCAULT'S PENDULUM: AN ELEMENTARY ANALYSIS.

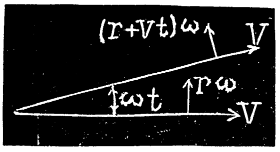
By W. H. JACKSON, Haverford College, Haverford, Pa.

The intrinsic interest in tangible evidences of the Earth's rotation makes it desirable to introduce the following results at as early a stage as is possible. This is the justification for the following elementary exposition.

1. General Method. Let a point move about a center with constant angular velocity ω, and recede from it radially with constant velocity v, and let it be initially at a distance r from the center.

After a time t the radial velocity is in a direction making an angle ωt with its initial direction. The component velocities along and perpendicular to the initial radius CP are therefore initially v, $r\omega$. After a time t they are, respectively,

$$v \cos \omega t - (r+v t)\omega \sin \omega t,$$
$$v \sin \omega t + (r+v t)\omega \cos \omega t.$$



To find the components of acceleration, we must

divide the component changes in velocity by t and proceed to the limit when this is small. Remembering that

$$\lim_{t \doteq 0} \left(\frac{1-\cos \omega t}{t}\right)=0, \quad \lim_{t \doteq 0} \left(\frac{\sin \omega t}{t}\right)=\omega,$$

we obtain along the radius and perpendicular to it, respectively,

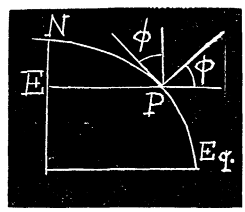
$$-r\,\omega^2, \quad 2\,v\,\omega. \tag{1}$$

A body moving with uniform velocity relative to the Earth would have on that account a certain acceleration relative to "fixed axes." This acceleration reversed will therefore be observed when the body moves freely on the Earth's surface.

The proof does not assume that the Earth is spherical. The latitude of any place is defined as the angle which the apparent direction of gravity makes with the plane of the equator. The latitude of a place is positive or negative according as its position lies to the north or south of this plane.

We shall consider the effect of motion along standard directions, and afterwards combine the results.

2. North-South Motion. Suppose that a body is moving with uniform velocity v from south to north. When at rest in a place P, of latitude ϕ, it has a uniform angular velocity ω about E, the projection of P on the Earth's axis. A velocity v, due north has a component $-v \sin \phi$ along EP, and the corresponding change in acceleration produced by this velocity is



$$-2\,\omega\,v \sin \phi \text{ by (1)},$$

in the direction of the Earth's rotation from west to east. Relative to the Earth, a body moving freely would experience an acceleration

$$2\,\omega\,v \sin \phi \text{ to the east.} \tag{2}$$

3. West-East Motion. Suppose that a body is moving with uniform velocity v from west to east. The only change in acceleration is due to the increase in angular velocity from ω to $(\omega+v/PE)$; it is equal to

$$-PE\left(\omega+\frac{v}{PE}\right)^2+PE.\omega^2 \text{ by (1)},$$

$$=-2\,\omega\,v\left(1+\frac{v}{2\,\omega\,PE}\right) \text{ along } EP. \tag{3}$$

If we regard the ratio which the velocity of the body relative to the Earth bears to that due to the Earth's rotation as negligible, we obtain a horizontal component of acceleration of the same magnitude as before,

$$2\,\omega\, v \sin \phi \text{ to the south.} \tag{4}$$

4. Any motion parallel to the Earth's surface. Any motion parallel to the Earth's surface may be compounded of a south-north and west-east motion. A body moving freely in any direction with velocity v on the Earth's surface will therefore possess an acceleration

$$2\,\omega\, v \sin \phi \tag{5}$$

in a direction obtained by a clockwise rotation of a right angle from its direction of motion.

5. Foucault's Pendulum. Suppose the bob of a simple pendulum is started so as to oscillate in any vertical plane, if r denotes its distance from its equilibrium position and v its velocity along that radius, the relative acceleration due to the Earth's rotation is

$$2\,\omega\, v \sin \phi.$$

But by (1), this is just the acceleration due to a rotation of angular velocity ($\omega \sin \phi$) about the vertical axis. The corresponding radial acceleratiod of $-r\,\omega^2 \sin^2 \phi$ is absent but that is negligible owing to the smallness of ω. That is, a pendulum started swinging in any vertical plane will rotate in the clockwise direction with uniform angular velocity

$$\omega \sin \phi. \tag{6}$$

6. The deflection of a body falling vertically. Suppose a body to move with uniform velocity v along EP, the change in its real acceleration is

$$2\,\omega\, v \text{ to the east by (1).}$$

A body moving freely along EP, therefore possesses a relative acceleration of

$$-2\,\omega\, v \text{ to the east.}$$

If a body falls freely with velocity v, this velocity may be resolved into

$$-v \sec \phi \text{ along } EP,$$

and $$-v \tan \phi \text{ to the north.}$$

Its resulting acceleration to the east is therefore

$$2\,\omega\, v(\sec \phi - \tan \phi \sin \phi).$$

That is, $$2\,\omega\, v \cos \phi \text{ to the east.} \tag{7}$$

The total displacement in a fall from a height h is therefore given approximately by writing $v=gt$, integrating with respect to t twice and substituting $t=(2\,h/g)^{\frac{1}{2}}$. The displacement is found to be

$$\tfrac{1}{3}\,\omega \cos \phi (8\,h^3/g^3)^{\frac{1}{2}}. \tag{8}$$

Finally, there is one component of relative acceleration which has not been written down; that is the vertical component due to the velocity v from west to east. On reference to equation (3), it is seen that this component is

$$2\,\omega\, v \cos \phi \text{ vertically upwards.} \tag{9}$$

7. *The general equations of relative motion.* If we take axes x, y, and z to be east, north, and vertically upwards, respectively, the preceding results are expressed by the usual equations for the motion of a body moving freely:

$$\begin{aligned} x'' &= 2\,y'\,\omega \sin \phi - 2\,z'\,\omega \cos \phi, && \text{by (2), (7).} \\ y'' &= -2\,x'\,\omega \sin \phi, && \text{by (4).} \\ z'' &= 2\,x'\,\omega \cos \phi, && \text{by (9).} \end{aligned}$$

ON THE REPRESENTATION OF NUMBERS AS THE SUM OF TWO SQUARES.

By M. KABA in Collaboration with L. E. DICKSON.

Consider the representations expressible as the sum of the squares of two numbers. Following Jacobi's notation for the theta-function with a special argument, we have

$$\theta(k) = \sqrt{\frac{2\,k}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + \ldots$$

And on the other hand,

$$\frac{2k}{\pi}=1+4\left[\frac{q}{1-q}-\frac{q^3}{1-q^3}+\ldots.\right]$$

Comparing these two results we have

$$(1) \qquad 1+4\left[\frac{q}{1-q}-\frac{q^3}{1-q^2}+\frac{q^5}{1-q^5}-\ldots.\right]=(1+2q+2q^4+2q^9+\ldots.)^2.$$

When we develop the terms of the left member, we get

$$1+4[(q+q^2+q^3+\ldots)-(q^3+q^6+q^9+\ldots) \\ +(q^5+q^{10}+q^{15}+\ldots)-\ldots]=1+4\Sigma N_e q^e.$$

We note that the sign preceding a series is $+$ or $-$, according as the first exponent is of the form $4m+1$ or $4m+3$. For

$$e=2^a p_1^{\pi_1} p_2^{\pi_2}\ldots p_n^{\pi_n} \quad (p_1, \ldots, p_n \text{ distinct odd primes}),$$

q^e occurs in those series, and only those, whose first exponents are of the form $p_1^{r_1}\ldots p_n^{r_n}$ $(0 \leqq r_i \leqq \pi_i)$. Let $e_i=+1$ or -1, according as p_i is of the form $4m+1$ or $4m+3$. But the product of two numbers each of the form $4m+1$, or each of the form $4m+3$, is of the form $4k+1$; while the product of $4m+1$ by $4l+3$ is of the form $4k+3$. Hence $p_1^{r_1}\ldots p_n^{r_n}$ is of the form $4m+1$ or $4m+3$ according as $e_1^{r_1}\ldots e_n^{r_n}=+1$ or -1. Thus

$$N_e=\Sigma\, e_1^{r_1} e_2^{r_2}\ldots e_n^{r_n} \quad (0 \leqq r_i \leqq \pi_i,\ i=1, \ldots, n),$$

$$(2) \qquad N_e=(1+e_1+e_1^2+\ldots+e_1^{\pi_1})(1+e_2+\ldots+e_2^{\pi_2})\ldots(1+e_n+\ldots+e_n^{\pi_n}).$$

But in view of the right member of (1), N_e is also the number of ways of representing e as the sum of the squares of two integers, zero or positive, provided we regard j^2+k^2 and k^2+j^2 as distinct representations when $j \neq k$, $j \neq 0$, $k \neq 0$ (whereas $0+s^2$ and s^2+0 give the same representation). *The number N_e of such representations of e is given by* (2).

In case e has a prime factor of the form $4m+3$, then $e_i=-1$, and $1+e_i+\ldots+e_i^{\pi_i}=0$ or 1, according as π_i is odd or even. Thus if π_i is odd there is no representation of e as the sum of two squares. If π_i is even, $N_e=N_{e'}$, where $e'=e/p_i^{\pi_i}$. If $N_{e'}=0$, there is again no representation of e. For $N_{e'}>0$, there are two representations $e'=x'^2+y'^2$ and hence representations $e=x^2+y^2$, $x=x'p_i^{\pi_i/2}$, $y=y'p_i^{\pi_i/2}$. Since $N_e=N_{e'}$, every representation of e may be derived from those of e' by multiplying the variables by $p_i^{\pi_i/2}$.

When the variables have a common factor >1, the representation is called improper.

Theorem. *There exists no representations as the sum of two squares for a number e having a prime factor of the form* $4m+3$ *occurring to an odd power; no proper representations when such a factor occurs to an even power. If* $P=p_1^{\pi_1}\ldots p_s^{\pi_s}$, *where* $p_1, \ldots, p_s$ *denote all the distinct primes of the form* $4m+3$ *which divide e, and if* $\pi_1, \ldots, \pi_s$ *are all even, there are as many improper representations of e as there are representations of e/P; every representation of e is of the type* $(P^{\frac{1}{2}}x)^2+(P^{\frac{1}{2}}y)^2$.

Examples. $N_e=0$ for $e=3, 7, 15, 21, 27, 63$; $N_9=1$, $N_{45}=2$, $N_{225}=3$, $45=3^2+6^2=6^2+3^2$, $9.25=15^2=9^2+12^2=12^2+9^2$.

The problem thus reduces to the case in which every prime factor of e is of the form $4m+1$. Then by (2),

$$N_e=(\pi_1+1)(\pi_2+1)\ldots(\pi_n+1). \tag{3}$$

For example, $N_5=N_{10}=2$, $N_{25}=3$, $25=5^2=3^2+4^2=4^2+3^2$.

When e is of the form $2^a p$, we have $N_e=2$. Removing the restriction that j^2+k^2 and k^2+j^2 shall be regarded as distinct representations, we obtain the well-known theorem of Fermat:

Every prime number p of the form $4m+1$ *(and every product* $2^a p$*) can be represented in one and only one way as the sum of two squares.*

In general, the representations enumerated in (3) include improper ones. For instance, if $\pi_1>1$, the $(\pi_1+1)(\pi_2+1)\ldots$ representations x^2+y^2 of e/p_1^2 yield improper representations $(p_1x)^2+(p_1y)^2$ of e. There are only 2^{n-1} distinct representations of e not distinguishing the order or signs of the variables in x^2+y^2 (Dirichlet-Dedekind, *Zahlentheorie*, p. 164).

A METHOD FOR CHANGING THE SCALE OF A NUMBER.

By C. E. WHITE, Nashville, Tennessee.

To convert a number in the scale s to a number in the scale $s-a$, where a may be positive or negative, the following process may be used.

Let $N=p_0p_1p_2p_3\ldots p_n=p_n+sp_{n-1}+s^2p_{n-2}+\ldots=f(s)$. By Taylor's theorem,

$$f(s)=f(a)+(s-a)f'(a)+\frac{(s-a)^2}{2!}f''(a)+\frac{(s-a)^3}{3!}f'''(a)+\ldots$$

Dividing $f(a)$ by $(s-a)$ we get $q_0+\dfrac{r_0}{s-a}$ where q_0 is the quotient and r_0 the remainder, or $f(a)=r_0+q_0(s-a)$.

Hence, $f(s)=r_0+(s-a)\,[f'(a)+q_0]+\frac{(s-a)}{2\,!}f''(a)+\ldots$

In like manner, $f'(a)+q_0=r_1+q_1(s-a)$ and $\frac{f''(a)}{2\,!}+q_1=r_2+q_2(s-a)$, and so on. Then

$$f(s)=r_0+r_1(s-a)+r_2(s-a)^2+r_3(s-a)^3+\ldots+r_n(s-a)^n,$$
and $N=r_nr_{n-1}\ldots r_3r_2r_1r_0$, in the scale $s-a$.

Let it be required, for example, to convert 567834 in the scale of 12 to the scale of 11.

$$f(s)=4+3s+8s^2+7s^3+6s^4+5s^5.$$

The value of the functions, and of the q's and r's are shown below.

$f(1)$,	$f'(1)$,	$\frac{f''(1)}{2\,!}$,	$\frac{f'''(1)}{3\,!}$,	$\frac{f^{\text{iv}}(1)}{4\,!}$,	$\frac{f^{\text{v}}(1)}{5\,!}$
33	89 3	109 8	81 10	31 8	5 3
0	4	7	3	6	8

Hence, $N=863740$ in the scale of 11.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

210. Proposed by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Simplify, $\log[\sqrt[3]{(137)}\,\sqrt[7]{(56)}\div\sqrt[10]{(187)}\,\sqrt[4]{(75)}]$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and A. H. HOLMES, Brunswick, Me.

Let P be the value of the expression in the parentheses.
Then $\log P=\frac{1}{3}\log 137+\frac{1}{7}\log 56-\frac{1}{10}\log 187-\frac{1}{4}\log 75$.

A similar example is given in Dickson's *College Algebra*, p. 20, ex. 6. Professor Dickson gives an answer in accordance with the above solution. It is our opinion, and in this opinion concur Professor E. R. Hedrick of the Missouri State University, and Professor George Melcher of the Missouri State Normal School of Springfield, that the last sign should be +. There is quite a general agreement among mathematical writers, that the operations of multiplication and division should be performed in the exact order of their occurrence. Thus $4\times6\div3\times2=16$ and not 4. However, Professor Dickson says, were one to ask for the log of $ab\div cd$ one would surely give as an answer, $\log a+\log b-\log c-\log d$, and to this form the numerical form in the problem corresponds. While we recognize the force of Professor Dickson's argument, yet we believe that when there is a possibility of ambiguity, every doubt should be removed by an explicit notation or else by following a well established usage.

312. Proposed by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.

Two roots of the cubic $x^3-px^2+qx-c=0$ are equal. Find their value in terms of p, q, and c.

I. Solution by GEORGE W. HARTWELL, University of Kansas, Lawrence, Kas.; V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.; and G. I. HOPKINS, Instructor in Mathematics and Astronomy, Manchester High School, N. H.

Let m, m, n be the three roots. Then

$$2m+n=p\ldots(1);\quad 2mn+m^2=q\ldots(2);\quad m^2n=c\ldots(3).$$

Solving (1) and (2) for m and n we have,

$$m=\frac{p\pm\sqrt{(p^2-3q)}}{3},\qquad n=\frac{p\mp2\sqrt{(p^2-3q)}}{3}.$$

Substituting these values in (3),

$$(2p^2-6q)\left(\frac{p\pm\sqrt{(p^2-3q)}}{3}\right)=pq-9c\ldots(4).\text{ But }\frac{p\pm\sqrt{(p^2-3q)}}{3}=m.$$

Therefore, from (4), $m=\dfrac{pq-9c}{2p^2-6q}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; S. LEFSCHETZ, Wilkinsburg, Pa.; and the PROPOSER.

Let $f(x)=x^3-px^2+qx-c=0$, $f'(x)=3x^2-2px+q=0$.

If $f(x)$ has two equal roots, $f'(x)$ contains one, and hence the greatest common divisor of $f(x)$ and $f'(x)$ gives one of the equal roots. Now if

$$(pq-9c)(27c+4p^3-15pq)=4q(3q-p^2)^2,\text{ or}$$
$$18cpq+p^2q^2-4cp^3-4q^3-27c^2=0,$$

then $2(3q-p^2)x+pq-9c$ is the greatest common divisor of $f(x)$ and $f'(x)$.

Therefore, $x=(9c-pq)/[2(3q-p^2)]$ is one of the equal roots.

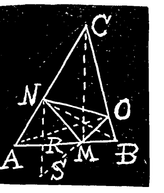
GEOMETRY.

339. Proposed by G. E. BROCKWAY, Boston, Mass.

Of all triangles that can be inscribed in a given triangle, that formed by joining the feet of the altitudes has the minimum perimeter. Prove by means of the straight line and circle.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Assuming the points N and O in the sides AC and BC, $NM+MO$ is a minimum, if $\angle NMA=\angle OMB$; for letting fall the perpendicular NR and extending it to S by its own length, OMS becomes a straight line. It follows from this that in the case of MNO being a triangle of minimum perimeter, $\angle NMA=\angle BMO=\alpha$, $\angle ANM=\angle CNO=\beta$, $\angle MOB=\angle NOC=\gamma$.



Now, $\alpha+\beta=180^\circ-A$, $\alpha+\gamma=180^\circ-B$, $\beta+\gamma=180^\circ-C$.

$\therefore \alpha+\beta+\gamma=\frac{1}{2}(540^\circ-180^\circ)=180^\circ$. $\therefore \alpha=C$, $\beta=B$, $\gamma=A$.

Consequently, the triangle MNO is the pedal triangle.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

When we wish to find a point P on a given line so that the sum of the distances $PR+PS$ to two given points, is a minimum, it is easy to show that PS and PR must make equal angles with the given line. Let M, L, K be the feet of the altitudes; M on AB, L on AC, K on BC. Then if M, L are fixed, K is the point on BC such that $KL+KM$ is a minimum, since KL, KM make equal angles with BC. Similarly, for M, K and L, K fixed in turn, respectively. $LM+LK$ is a minimum and $ML+MK$ is a minimum for the reason cited above.

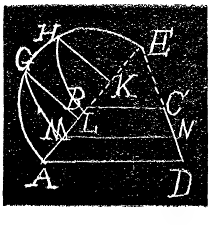
This seems the easiest and simplest proof.

340. Proposed by J. H. MEYERS, S. J., Sacred Heart College, Augusta, Ga.

Given trapezoid $ABCD$. Prolong AB and CD, the non-parallel sides, to meet in E. On AE as diameter construct semi-circle $AHGE$. With BE as radius construct arc BG. Draw GK perpendicular to AE. Bisect AH at L. Erect KH perpendicular to AE. Construct arc HM with HE as radius. Draw MN perpendicular to DC. Prove that MN bisects the trapezoid $ABCD$, angles ADC and BCD being right angles.*

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

We may generalize this theorem by drawing the line MN parallel to AD, instead of perpendicular to DC.



$\triangle ADE : \triangle EMN=AE^2 : EM^2=AE^2 : EH^2=AE^2 : AE\times EL=AE : EL$.

$\therefore \triangle ADE-\triangle EMN : \triangle EMN=AE-EL$, or $AMND : \triangle EMN=AL : EL\ldots$(I).

$\triangle EMN : \triangle EBC=ME^2 : :BE^2=KE^2 : GE^2=AE\times EL : AE\times EK=EL : EK$.

$\therefore \triangle EMN-\triangle EBC : \triangle EMN=EL-EK : EL$, or $MBCN : \triangle EMN=LK : EL\ldots$(II).

Comparing (I) and (II), $AMND : MBCN=AL : LK=1 : 1$.

*The reading of this problem has been slightly changed to correspond to the figure. ED. F.

$\therefore AMND=MBCN$.

Note. If the angles at C and D are right, MN, of course, is perpendicular to CD. As the proposer states this theorem, it is too restricted.

Also solved by G. B. M. Zerr, J, A. Caparo, A. H. Holmes, and analytically by V. M. Spunar.

341. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Given $\rho=\cos(m/n)\theta$, where m and n are integers without a common factor. Deduce rules for finding by inspection:

(1) The angle between the beginning and end of any loop of this curve;
(2) The number of distinct loops. [A loop is a portion of the curve between consecutive zero radii vectores.]

Solution by the PROPOSER.

(1) Consecutive values of $\frac{m}{n}\theta$ which make ρ vanish must differ by π. Hence, consecutive values of θ differ by $n\pi/m$.

(2) Since ρ is a periodic function of θ, the period being $\frac{2n\pi}{m}$, the rectangular coordinates have the period $2n\pi$. The whole figure being repeated with that period, attention may be confined to the interval $\theta=\frac{n}{m}\frac{\pi}{2}$ to $\theta=\frac{n}{m}\frac{\pi}{2}+2n\pi$. In this interval there are $2m$ loops, as follows from (1). It remains to be seen when these loops will be all distinct.

The coincidence of two loops occurs only when a loop in which ρ is negative repeats one in which ρ is positive. In that case, any point (ρ_1, θ_1) on one loop coincides with $[-\rho_1, \theta_1+\pi(2\lambda+1)]$ on the other [λ, some integer]. But, since $-\rho_1=a\cos\frac{m}{n}[\theta_1+\pi(2\lambda+1)]$, and $-\rho_1=-a\cos\frac{m}{n}\theta_1$, it is clear that $\frac{m}{n}(2\lambda+1)\pi$ is an odd multiple of π, say $\frac{m}{n}(2\lambda+1)=2k+1$. This is impossible if either m or n is even. Moreover, if both are odd, λ and k can always be found to satisfy the condition; *e. g.*, $\lambda=\frac{n-1}{2}$, $k=\frac{m-1}{2}$. Each loop for $\rho<0$ will then repeat a loop for $\rho>0$. Hence the number of distinct loops is (a) $2m$ if either m or n is even, (b) m if both m and n are odd.

Remark. The curve $\rho=\sin\frac{m}{n}\theta$ is simply the foregoing curve rotated through the angle $-\frac{n}{m}\frac{\pi}{2}$. Hence the conclusions reached above apply equally to this curve.

Also solved by G. B. M. Zerr, V. M. Spunar, nad J. W. Clawson, a Sophomore in Williams College.

CALCULUS.

269. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Prove that $\int_0^1 (x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{dx}{x}=\frac{\pi}{a}\tan(\frac{1}{2}\pi a)$.

Solution by S. A. COREY, Hiteman, Iowa.

Developing $(x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{1}{x}$ into a power series in x and integrating, we have

$$\int (x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{dx}{x}=2\left[\frac{x^{1+a}}{1+a}+\frac{x^{1-a}}{1-a}+\frac{x^{3+a}}{3(3+a)}+\frac{x^{3-a}}{3(3-a)}+\dots\right]$$

and the definite integral of the problem becomes equal to the series,

$$2\left[\frac{1}{1+a}+\frac{1}{1-a}+\frac{1}{3}\cdot\frac{1}{3+a}+\frac{1}{3}\cdot\frac{1}{3-a}+\dots\right]$$

which reduces to the form,

$$4\left[\frac{1}{1^2-a^2}+\frac{1}{3^2-a^2}+\frac{1}{5^2-a^2}+\dots\right]. \tag{1}$$

Fourier's cosine series for $\cos(ax)$ is

$$\cos(ax)=\frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2}+\frac{\cos x}{1^2-a^2}-\frac{\cos 2x}{2^2-a^2}+\frac{\cos 3x}{3^2-a^2}-\dots\right], \tag{2}$$

where a is fractional. When $x=\pi$, (2) gives

$$\cos(a\pi)=\frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2}-\frac{1}{1^2-a^2}-\frac{1}{2^2-a^2}-\frac{1}{3^2-a^2}-\dots\right], \tag{3}$$

and when $x=0$, it gives

$$1=\frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2}+\frac{1}{1^2-a^2}-\frac{1}{2^2-a^2}+\frac{1}{3^2-a^2}-\dots\right]. \tag{4}$$

Subtracting (3) from (4), we have

$$1-\cos(a\pi)=\frac{4a\sin(a\pi)}{\pi}\left[\frac{1}{1^2-a^2}+\frac{1}{3^2-a^2}+\frac{1}{5^2-a^2}+\ldots\right], \quad (5)$$

whence the value of the series, (1), when $0<a>1$, is readily found to be $\frac{\pi}{a}\tan\left(\frac{\pi a}{2}\right)$, as required by the problem.

When $a>1$, or $a<-1$, let $b=a^{-1}$, and substitute in the problem. The definite integral does not change its form by making this substitution, but the right hand member of the equation does change. Hence the given equation does not hold for values of $a>1$, or <-1. It can readily be shown that the equation holds when $a=0$, $a=\pm1$, and when $0>a>-1$. The range of values of a in which the given equation holds is, then, $-1 \overline{\overline{<}} a \overline{\overline{<}} 1$.

Also solved by C. N. Schmall, V. M. Spunar, J. Scheffer, and the Proposer.

270. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum\limits_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}=\frac{\pi}{2a^{2n-1}}.\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\ldots.\frac{(2n-3)}{(2n-2)}.\frac{1}{2a^{2n}}$, n being a positive integer >1.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; C. N. SCHMALL, New York City; and J. SCHEFFER, A. M., Hagerstown, Md.

Assuming $\sum\limits_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}\equiv\int_0^\infty\frac{dx}{(a^2+x^2)^n}$, the integral may be evaluated in two different ways.

(1) Assume the equation

$$\int_0^\infty\frac{dx}{x^2+a}=\frac{\pi}{2}.\frac{1}{a^{\frac{1}{2}}}.$$

Now differentiate both sides $(n-1)$ times with regard to a, we obtain

$$\int_0^\infty\frac{dx}{(x^2+a)^n}=\frac{\pi}{2}\,\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\,\frac{1}{a^{n-\frac{1}{2}}};$$

and substituting a^2 for a, we have

$$\int_0^\infty\frac{dx}{(a^2+x^2)^n}=\frac{\pi}{2}.\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}.\frac{1}{a^{2n-1}}.$$

SCHMALL.

(2) Or, by putting $x=a\tan\theta$ in the given integral, we have

$$\int_0^\infty\frac{dx}{(a^2+x^2)^n}=\frac{1}{a^{2n-1}}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2n-2}d\,\theta=\frac{1}{a^{2n-1}}.\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}.\frac{\pi}{2}.$$

SCHMALL, ZERR, SCHEFFER.

MECHANICS.

222. Proposed by W. J. GREENSTREET, Stroud, England.

Find the maximum angle of inclination to the line of greatest slope of a uniform rod resting on a rough inclined plane and capable of turning freely round a point on it.

II. Solution by the PROPOSER.

Let the point on the rod about which it turns be distant x and y from the ends $(x>y)$. Let α be the inclination of the plane, and θ the angle the rod in its initial position makes with the greatest line of slope. Let W be the weight per unit length of the rod. Consider the normal reactions as acting at the mid points, respectively, of AO, OB, where AB is the rod and O the pivot. μ is the coefficient of friction.

Resolving perpendicular to the plane for each part AO, OB,

$$R=\frac{x}{2}W\cos\alpha,\qquad R'=\frac{y}{2}W\cos\alpha.$$

There is an unknown force at O. Take moments at O,

$$\mu.\frac{x}{2}R-\frac{x^2}{4}W\sin\alpha\sin\theta+\mu.\frac{y}{2}R'+\frac{y^2}{4}W\sin\alpha\sin\theta=0,$$

and substituting for R, R',

$$\mu\frac{x^2}{4}.W\cos\alpha+\mu\frac{y^2}{4}W\cos\alpha=\frac{W\sin\alpha\sin\theta}{4}(x^2-y^2),$$

$$\mu\cot\alpha\,(x^2+y^2)=\sin\theta\,(x^2-y^2).$$

$$\therefore\sin\theta=\mu\cot\alpha\frac{x^2+y^2}{x^2-y^2},\qquad\theta=\sin^{-1}\left(\mu\cot\alpha.\frac{x^2+y^2}{x^2-y^2}\right).$$

224. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A steel clock spring $w=\frac{7}{8}$ inch wide, $t=\frac{1}{32}$ inch thick, is wound around an axle $d=\frac{1}{4}$ inch in diameter. Find the greatest available moment for running the clock, using a factor of safety $f=6$.

Solution by the PROPOSER.

Let $OH=r=$radius of curvature; $HL=y=$distance of any fiber from the gravity axis HH; $x=$length LM of this fiber; $MS=dx$; $RH=z=$distance between the neutral axis and the gravity axis; $\angle MRS=\phi$, $\angle AOB=\theta$; $s=$

From (1), $z=.00066$. $s=6416.9M$. The ultimate strength of hard steel is 240000 lb./in.^{2}.

$\therefore s=240000/f=240000/6=40000$ lb./in.^{2}.

$\therefore 40000=6416.9M$. $\therefore M=6.2335$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

159. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that if the equation $y^3=2x^2-1$ be possible in integers, $y=24n^2-1$, or $2n^2-1$, and find three solutions.

Solution by the PROPOSER.

The equation may be written $(y+1)(y^2-y+1)=2x^2$.

Since y^2-y+1 is always odd, it is evident that $y+1$ must be even. Since $y^2-y+1=(y+1)(y-2)+3$ it is evident that $y+1$ and y^2-y+1 can have no common factor but 3. Therefore we have the following possibilities for y: $y=2\times3m^2-1$, or $y=2n^2-1$.

Since y^3 is represented by the form $2x^2-1$, 2 must be a quadratic residue of y. Therefore $y=8a\pm1$, and this is possible in the first expression only when $m=2n$. Then either $y=24n^2-1$ or $2n^2-1$.

Substituting these values of y in the original equation, we have

$$192n^4-24n^2+1=r^2 \text{ or } 4n^4-6n^2+3=r^2.$$

The first equation has the solution $n=0$ and 1 which give $y=-1$, $x=0$; $y=23$, $x=78$.

The second equation has the solution $n=1$, which gives $y=1$, $x=1$.

AVERAGE AND PROBABILITY.

198. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Find the average length of a hole at random through a given cylinder.

No solution of this problem has been received.

199. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A circle is inscribed in a given square. Two points are taken at random within the square but without the circle. What is the chance the distance between the points does not exceed the side of the square?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

This is the same as 196, but as there is a distance less than the side of the square when both points are taken one each in opposite corners, it is

desirable that another solution be presented. Professor Carmichael called my attention to this error.

As a solution by Calculus would be very long and tedious and well nigh impossible, we will use a formula given in the *Encyclopedia Britannica.* It is as follows:

If p is the probability of a certain condition being fulfilled by n points within an area A, p' the probability when they fall on area $A+B$ (B without A), p_1 the probability when one point falls on B and the rest on A, then $(p'-p)A=nB(p_1-p)$.

In the problem, $A=\pi a^2$, $B=(4-\pi)a^2$, $n=2$, where $2a$=side of the square. Now $p'=\pi-\frac{13}{6}$ (MONTHLY, Vol. III, No. 11, page 285); $p=1$ for both points on A.

$\therefore (\pi-\frac{13}{6}-1)\pi a^2=2(4-\pi)(p_1-1)a^2$.

$\therefore p_1=\dfrac{6\pi^2-31\pi+48}{12(4-\pi)}$, the probability that the distance is less than the side of the square when one point is within, the other without the circle.

Similarly, $(p'-P)B=2A(p_1-P)$.

$\therefore P=\dfrac{2\pi p_1-p'(4-\pi)}{3\pi-4}$. $P=\dfrac{15\pi^2-76\pi+104}{3(4-\pi)(3\pi-4)}$.

MISCELLANEOUS.

176. Proposed by WM. E. HEAL, Coffeyville, Kansas.

In Grassman's *Extensive Algebra*, $e_1e_2=-e_2e_1$. If $e_1=e_2$, $e_1^2=-e_1^2=0$. In quaternions, $ij=-ji$, $i^2j=i.ij=ik=-j$, $i^2=-1$. Reconcile these apparently divergent results.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

i, j, k are unit vectors at right angles to each other. By definition: The effect of any unit vector acting as a multiplier upon another at right angles to it, is the turning of the latter in a positive direction (counter clock wise) in a plane perpendicular to the operator or multiplier through an angle $\frac{1}{2}\pi$. Hence the product or quotient of two unit vectors at right angles is a unit vector perpendicular to their plane. In a positive direction i operating on j produces k, or $ij=k$. Similarly, $jk=i$, $ki=j$. j operating on i in a negative direction produces $-k$. Therefore, $ji=-k$. Similarly, $ik=-j$, $kj=-i$.

The operation of multiplying here is not a numerical product, and hence it is a geometric multiplication, and not an algebraic process.

As the effect of i, j, k as operators is to turn a line from one direction into another which differs from it by 90°, they are called quadrantal versors.

Now $ij=k$, $ik=-j$. $\therefore i.ij=-j=-1.j=i^2j$.

$\therefore i.i=i^2=-1$. Also, $ij=k$, $ji=-k$. $\therefore ij=-ji$.

(See Hardy's *Elements of Quaternions*, pp. 40-48.)

PROBLEMS FOR SOLUTION.

ALGEBRA.

318. **Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.**

Sum to infinity the series $n/(4n^2-1)^2$ beginning with $n=1$.

319. **Proposed by C. N. SCHMALL, New York City.**

A man desires to purchase eggs at 5 cents, 1 cent, and $\frac{1}{2}$ cent, respectively, in such numbers that he will obtain 100 eggs for a dollar. How many solutions in rational integers?

320. **Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.**

Solve for t, $\cos t=m\cos 2t$.

GEOMETRY.

345. **Proposed by LLOYD HOLSINGER, Bradley Polytechnic Institute, Peoria, Ill.**

If a variable polygon move in such a way that its n sides turn severally round n fixed points O_1, O_2, ..., O_n while $n-1$ of its vertices slide, respectively, along $n-1$ fixed straight lines v_1, v_2, ..., v_{n-1}, then the last vertex will describe a conic; and the locus of the point of intersection of any pair of non-adjacent sides will also be a conic. Cremona's *Projective Geometry.*

346. **Proposed by G. I. HOPKINS, M. A., Professor of Mathematics and Astronomy, High School, Manchester, N. H.**

Prove the theorem for finding the lateral area of a frustum of a cone without the use of the theory of limits.

347. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

ABC is a triangle, and D, E, F are the mid points of the arcs of its nine-point circle cut off by BC, CA, AB, respectively. The inscribed circle touches these sides at X, Y, Z. Are the lines DX, EY, FZ concurrent? A purely geometrical discussion required.

CALCULUS.

276. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

In a certain country the tax per \$1 on a person's income varies as the cube root of the number of dollars, and when the income is \$8000 the rate per dollar is 5 cents. Find the largest net income possible.

277. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

Find $\dfrac{d^2x}{ds^2}$ and $\dfrac{d^2y}{ds^2}$ for $y=c\sinh\dfrac{x}{c}$.

278. **Proposed by S. A. COREY, Hiteman, Iowa.**

If C be Euler's constant, .577,215,664,9... and if B_1, B_2, B_3, etc., be Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc., prove that

$$C=\tfrac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-\frac{B_4}{8}+\ldots-(1)^m\frac{B_m}{2m}+\ldots.$$

MECHANICS.

231. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A body is projected towards the Earth's center with a velocity from infinity at a distance $a=2r$ from the center. If the Earth were an airless sphere with a radius equal to its present mean radius, and gravity equal to its present intensity, and having a hole from surface to center, with what velocity and in what time would it arrive at the center.

232. Proposed by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.

Given, the diameter d of a gas engine cylinder, length of connecting rod l, stroke s, velocity of crank pin v, length of exhaust port equal to one-third the circumference of the cylinder. Find the average height h of exhaust port opened during one cycle if the port is fully opened, when the piston is at its lowest position.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

163. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Prove that the equation $y^n=mx+1$ always has at least one positive integer solution (different from $y=1$, $x=0$), whatever integer values m and n may have.

164. Proposed by G. J. GRIFFITHS, M. A., in Educational Times (Unsolved).

Prove that the sum of the squares of the reciprocals of all integers which are not divisible by the square of any prime is $15/\pi^2$.

NOTES AND NEWS.

Mr. H. P. Kean, Assistant in Mathematics at the University of Illinois, has been elected Professor of Mathematics in Ripon College. ED. M.

Will any reader who reads this note help me to find a copy of the *Mathematical Companion* edited and published by John D. Williams? This journal was started in 1828 and was published until 1831. Bolton, in his catalog of scientific journals, says there is a copy of this journal in the Library of Harvard University. Professor Byerly informed me that he had the librarian make a careful search for it with the result that a sort of prospectus of it was all that could be found. Also who has a complete set of Harvell's *Messenger of Mathematics?* ED. F.

Courses in Mathematics for the year 1909-10, University of Pennsylvania. By Professor E. S. Crawley: Solid analytic geometry, two hours; Higher plane curves, three hours; Mathematics of insurance, two hours. By Professor G. E. Fisher: Advanced calculus, two hours; Calculus of variations, two hours. By Professor I. J. Schwatt: Infinite series and products, two hours; Definite integrals, three hours. By Professor G. H. Hallett: Modern higher algebra, three hours (first half year); Galois theory of equations, three hours (second half year); Theory of groups of a finite order, three

hours; Lie's theory of continuous groups, three hours (first half year). By Professor F. H. Safford: Mathematical theory of precision of measurements, three hours (first half year); Curvilinear coordinates, three hours (second half year). By Dr. O. E. Glenn: Invariants and covariants, three hours.

Dr. B. F. Finkel has almost completed his History of American Mathematical Journals. This history gives a detailed account of the earliest American mathematical journals. As these journals are very rare, and inaccessible to the general reader, some of the most important articles in the earlier journals are reproduced in their entirety in this history. Thus, Adrain's discovery of the "Law of Least Squares" in the *Analyst*, 1804, and Professor Benjamin Pierce's proof that "No odd number can be a perfect number," *The Mathematical Diary*, 1825, are given in full in this history. In book form, the History would comprise about 300 pages, and would be the most complete history of American mathematical journals ever written.

After carefully going over the whole matter I have concluded to undertake the publication of this work, provided I can be assured of not losing too much money in the enterprise. I do not wish to make anything above the expense of publication. I have therefore concluded to publish 400 copies of the work, each copy to be numbered, and to sell them at $2.50 each. Persons wishing to secure a copy of this book will please give me their orders at once, so that in case I get enough orders to encourage the undertaking, I may begin work immediately.

The volumes will be numbered and signed in the order in which the subscriptions are received.

S. A. DIXON, Printer.

Springfield, Missouri.

BOOKS.

Azimuth. By George L. Hosmer, Assistant Professor of Civil Engineering, Massachusetts Institute of Technology. 16mo, v+73 pages, 6 figures. Morocco, $1.00 (4/6 net). New York: John Wiley & Sons.

The purpose of this volume is to present in compact form certain approximate methods of determining the true bearing of a line, together with the necessary rules and tables arranged in a simple manner so that they will be useful to the practical surveyor. It is a handbook rather than a textbook, hence many subjects have been wholly omitted which are ordinarily included in books on Practical Astronomy but which are not essential in learning to make the observations described in this book. In all of the methods here treated the object sought is to secure sufficient accuracy for the purpose of checking the measured angles of a survey with the least expenditure of time. For this reason many approximations have been made and many refinements omitted which simplify the calculations without introducing serious error into the results, and although such a treatment would scarcely be proper in a textbook the gain in simplicity and convenience would seem to justify its use in a book of this character. *Preface.*

THE AMERICAN MATHEMATICAL MONTHLY.

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VOL. XVI. JUNE-JULY, 1909. NOS. 6-7.

ON THE EXTENSION OF THE EXPONENTIAL THEOREM.*

By E. D. ROE, JR., Syracuse University.

§1. INTRODUCTION.

In our algebra,† we define

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{x}{n}\right)^n = F(x),$$

for all values of x and n real or complex, and denote by $f(x)$ the value which $F(x)$ takes when n is a positive integer and x is real. We prove that

$$f(x)=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots=e^x.$$

Next we prove that $F(x)=f(x)$ for any real value of x, in whatever way n as a real number becomes indefinitely great. Then we prove for a positive integral n and a complex x, which is called z, that $F(z)=f(z)$. Then we state, p. 240: "The extension to any real value of n may be made by considering the limits of the moduli and arguments of

$$(1+\frac{z}{m+1})^m,\quad (1+\frac{z}{n})^n,\quad (1+\frac{z}{m})^{m+1},$$

and finally extending to a negative n."

It is the object of the present paper to furnish the details of this statement.

*Presented to the American Mathematical Society, April 24, 1909.

†*College Algebra,* Metzler, Roe and Bullard (Longmans, Green and Co.), 1908.

§2. EXTENSION TO ANY POSITIVE n.

For convenience we divide the proof into three cases:

1. Where $90^\circ \geqq \arg z > 0$,
2. Where $180^\circ > \arg z > 90^\circ$,
3. Where $360^\circ > \arg z > 180^\circ$.

§3. CASE 1. $90^\circ \geqq$ ARG $z > 0$.

Here $z = \dfrac{MP}{OM} = \dfrac{MP}{1}$. And m being any integer we may always have

$$m+1 > n > m$$

and since $z = r(\cos\phi + i\sin\phi)$ we have, Fig. 1,

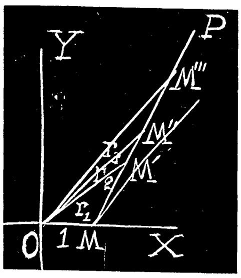

Fig. 1.

$$\frac{z}{m+1} = \frac{MM'}{OM} = \frac{MM'}{1} = \frac{r}{m+1}(\cos\phi + i\sin\phi),$$

$$\frac{z}{n} = \frac{MM''}{OM} = \frac{MM''}{1} = \frac{r}{n}(\cos\phi + i\sin\phi),$$

$$\frac{z}{m} = \frac{MM'''}{OM} = \frac{MM'''}{1} = \frac{r}{m}(\cos\phi + i\sin\phi),$$

where length $MM' = \dfrac{r}{m+1}$, length $MM'' = \dfrac{r}{n}$, length $MM''' = \dfrac{r}{m}$. We also have

$$1 + \frac{z}{m+1} = \frac{OM'}{1}, \quad \mathrm{mod}\left(1 + \frac{z}{m+1}\right) = \text{length } OM' = r_1,$$

$$MOM' = \arg\left(1 + \frac{z}{m+1}\right) = \phi_1,$$

$$1 + \frac{z}{n} = \frac{OM''}{1}, \quad \mathrm{mod}\left(1 + \frac{z}{n}\right) = \text{length } OM'' = r_2, \quad MOM'' = \arg\left(1 + \frac{z}{n}\right) = \phi_2,$$

$$1 + \frac{z}{m} = \frac{OM'''}{1}, \quad \mathrm{mod}\left(1 + \frac{z}{m}\right) = \text{length } OM''' = r_3, \quad MOM''' = \arg\left(1 + \frac{z}{m}\right) = \phi_3.$$

And if $90^\circ \geqq \arg z > 0$, $r_1 < r_2 < r_3$, $\phi_1 < \phi_2 < \phi_3$, whence

$$r^m{}_1 < r^n{}_2 < r^{m+1}{}_3, \quad m\phi_1 < n\phi_2 < (m+1)\phi_3;$$

that is

$$\text{mod } (1+\frac{z}{m+1})^m < \text{mod}(1+\frac{z}{n})^n < \text{mod } (1+\frac{z}{m})^{m+1}, \quad (1)$$

$$\arg (1+\frac{z}{m+1})^m < \arg (1+\frac{z}{n})^n < \arg (1+\frac{z}{m})^{m+1}. \quad (2)$$

Now $\underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m+1})^m = \underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m})^{m+1}$, since

$$\underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m+1})^m = \frac{\underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m+1})^{m+1}}{\underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m+1})} = f(z), \text{ and}$$

$$\underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m})^{m+1} = \underset{m \doteq \infty}{\mathrm{L}} (1+\frac{z}{m})^m (1+\frac{z}{m}) = f(z).$$

Hence, since when two complex numbers are equal their moduli and arguments are equal,

$$\underset{m \doteq \infty}{\mathrm{L}} \text{mod}(1+\frac{z}{m+1})^m = \underset{m \doteq \infty}{\mathrm{L}} \text{mod}(1+\frac{z}{m})^{m+1} = \text{mod } f(z),$$

$$\underset{m \doteq \infty}{\mathrm{L}} \arg (1+\frac{z}{m+1})^m = \underset{m \doteq \infty}{\mathrm{L}} \arg (1+\frac{z}{m})^{m+1} = \arg f(z).$$

Therefore, from the preceding inequalities, since $\text{mod}(1+\frac{z}{n})^n$ lies between two numbers, both of which have the same limit mod $f(z)$

$$\underset{n \doteq \infty}{\mathrm{L}} \text{mod } (1+\frac{z}{n})^n = \text{mod } f(z).$$

Similarly, $\underset{n \doteq \infty}{\mathrm{L}} \arg (1+\frac{z}{n})^n = \arg f(z).$

Therefore, $\underset{n \doteq \infty}{\mathrm{L}} (1+\frac{z}{n})^n = f(z).$

§4. CASE 2. $180^\circ > \text{ARG}\, z > 90^\circ$.

With the same notation as before:

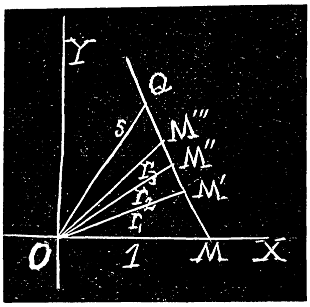

Fig. 2.

$$\phi_1 < \phi_2 < \phi_3, \quad m\phi_1 < n\phi_2 < (m+1)\phi_3,$$

and as before:

$$\underset{n \doteq \infty}{\text{L}} \arg(1+\frac{z}{n})^n = \arg f(z).$$

But we can not be sure that the inequalities (1) hold. We can however proceed as follows: By Trigonometry, Fig. 2, where length $MQ = \frac{r}{\lambda}$,

$$[\text{mod}(1 + \frac{z}{\lambda})]^2 = s^2 = 1^2 + (\frac{r}{\lambda})^2 + \frac{2r\cos\phi}{\lambda} = 1 + \frac{r}{p}, \text{ where } p = \frac{\lambda}{2\cos\phi + r/\lambda}.$$

$[\text{mod}(1+\frac{z}{\lambda})^\sigma]^2 = [(1+\frac{r}{p})^p]^{\sigma/p}$. If now σ and p are two real numbers which become indefinitely great in any way such that the difference between σ and λ remains finite then we have:

$$\underset{\lambda \doteq \infty}{\text{L}} (\frac{\sigma}{\lambda}) = 1. \quad p \doteq \infty, \text{ as } \lambda \doteq \infty. \quad \underset{\lambda \doteq \infty}{\text{L}} \frac{\sigma}{p} = 2\cos\phi.$$

$$\underset{\lambda \doteq \infty}{\text{L}} [\text{mod}(1+\frac{z}{\lambda})^\sigma]^2 = e^{2r\cos\phi}.$$

Taking the square root we have:

$$\underset{\lambda \doteq \infty}{\text{L}} \text{mod}(1+\frac{z}{\lambda})^\sigma = e^{r\cos\phi} = e^{\text{mod}(z)\cos\arg(z)}.$$

a result that is independent of λ and σ.

As special cases, if (λ, σ) have the pairs of values $(m+1, m)$, (n, n), $(m, m+1)$, (m, m), we see that the three numbers of the inequalities (1) have the same limit, and that this limit is $\text{mod}\, f(z)$.

Since the modulus and argument of $(1+\frac{z}{n})^n$ have for limit, respectively, the modulus and argument of $f(z)$, we have:

$$\underset{n \doteq \infty}{\text{L}} (1+\frac{z}{n})^n = f(z).$$

This method might also have been used in case 1.

§5. CASE 3. WHEN $360^\circ > \text{ARG } z > 180^\circ$.

We may fold over the figure on the axis of x, and consider the conjugate. Thus by cases 1 or 2:

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{z'}{n}\right)^n = f(z'),$$

where z' is the conjugate of z; *i. e.*,

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{x-yi}{n}\right)^n = f(x-yi).$$

As the real and the imaginary parts are respectively equal on the two sides of this equality, the equality will remain if i is changed into $-i$. Hence

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{x+yi}{n}\right)^n = f(x+yi), \text{ or } \underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{n}\right)^n = f(z).$$

§6. THE EXTENSION TO A NEGATIVE n.

Let $n = -p$, then

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{n}\right)^n = \underset{p \doteq \infty}{\mathrm{L}} \left(1-\frac{z}{p}\right)^{-p} = \underset{p \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{p-z}\right)^p$$

$$= \underset{p \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{\rho(\cos\alpha+i\sin\alpha)}\right)^p = \underset{p \doteq \infty}{\mathrm{L}} \left[\left(1+\frac{z(\cos\alpha-i\sin\alpha)}{\rho}\right)^\rho\right]^{p/\rho}$$

$$= f\left[z\mathrm{L}(\cos\alpha - i\sin\alpha)\right] = f(z), \text{ since}$$

$$p-z = p-x-yi = \rho(\cos\alpha+i\sin\alpha), \quad \rho = [(p-x)^2+y^2]^{\frac{1}{2}},$$

$$\underset{p \doteq \infty}{\mathrm{L}} \frac{p}{\rho} = 1, \quad \underset{p \doteq \infty}{\mathrm{L}} \alpha = 0, \text{ and } \underset{p \doteq \infty}{\mathrm{L}} (\cos\alpha - i\sin\alpha) = 1.$$

Hence in all cases when z is complex and n is real

$$F(z) = f(z).$$

§7. THE CASE WHEN z AND n ARE BOTH COMPLEX.

We quote from the text p. 240: "If both z and n are complex, we have, if $n=m(\cos\phi+i\sin\phi)$,

$$\underset{n\doteq\infty}{\mathrm{L}}\left(1+\frac{z}{n}\right)^n=\underset{m\doteq\infty}{\mathrm{L}}\left(1+\frac{z}{m(\cos\phi+i\sin\phi)}\right)^{m(\cos\phi+i\sin\phi)}$$

$$=\underset{m\doteq\infty}{\mathrm{L}}\left[\left(1+\frac{z(\cos\phi-i\sin\phi)}{m}\right)^m\right]^{\cos\phi+i\sin\phi}=\{f[z(\cos\phi-i\sin\phi)]\}^{\cos\phi+i\sin\phi}.$$

This is as far as we can carry the proof. If, however, we agree to give to a complex exponent such an interpretation that the third property, viz: $f(z)=[f(1)]^z$, shall still hold even when z is complex, we have

$$F(z)=\{f[z(\cos\phi-i\sin\phi)]\}^{\cos\phi+i\sin\phi}=f(1)^z=f(z).$$

Thus for all values of x and n, we have

$$F(x)=f(x).$$

That the series denoted by $f(x)$ is convergent has been seen from the mode of its derivation, since each of the constituent series of which it is composed is convergent whether x be real or complex. The result

$$\underset{n\doteq\infty}{\mathrm{L}}\left(1+\frac{x}{n}\right)^n=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

is known as the Exponential Theorem."

SYRACUSE UNIVERSITY, *12 April, 1909.*

RATIONAL EDGED CUBOIDS WITH EQUAL VOLUMES AND EQUAL SURFACES.

By L. E. DICKSON, The University of Chicago.

1. In the BULLETIN, May, 1909, p. 401, Professor Kasner proposed the problem to find two cuboids (rectangular parallelopipeds) with equal volumes and equal surfaces, and in which the dimensions are all integral.

The problem is to find two distinct triples of integers such that

$$(1) \qquad xyz=x'y'z', \quad xy+xz+yz=x'y'+x'z'+y'z'.$$

I shall prove that each integer must exceed unity and that the volume xyz must be the product of five or more primes. Among the solutions which I obtain, the simplest are

$$(2) \quad p,\ p,\ \tfrac{1}{2}(p+1)p^2; \quad p^2,\ p^2,\ \tfrac{1}{2}(p+1) \qquad (p \text{ odd}),$$

$$(3) \quad p,\ (p+1)q,\ (p+1)(p+1-q); \quad p+1,\ (p+1)p,\ q(p+1-q) \quad (1<q<p),$$

in which $q=r$ and $q=p+1-r$ give the same sets. While for (3) the volume contains at least six primes, for (2) there may be only five primes, for example,

(2′) 3, 3, 18; 9, 9, 2. 5, 5, 75, 25, 25, 3.

The simplest examples under (3) are

(3′) 3, 8, 8; 4, 12, 4. 4, 10, 15; 5, 20, 6.

Generalizations of (2) and (3) are given by (9)-(10) and (16)-(17).

The only solutions in which $xyz<200$ are the first solutions under (2′) and (3′).

In §§7-9, I show that for three rational edged cuboids with equal volumes and equal surfaces, the volume is the product of six or more primes, that no two of the triples are of the type $p_1, p_2, p_3p_4p_5p_6$, and that the cases in which two of the triples are $p_1, p_2, p_3p_4p_5p_6$ and $p_3, p_1p_2, p_4p_5p_6$ are excluded.

2. Theorem. *Two triples are identical if they have an element in common.*

If $x=x'$, equations (1) reduce to $yz=y'z'$, $y+z=y'+z'$.

3. Theorem. *In any solution each integer exceeds unity.*

Two sets with equal products and having an element unity must be of the form

$$(4) \qquad 1,\ f_1f_2f_3,\ f_4f_5f_6;\ f_1f_4,\ f_2f_5,\ f_3f_6.$$

Denote the corresponding surfaces by $2S_1$ and $2S_2$. Then

$$S_1=f_1f_2f_3+f_4f_5f_6+f_1\ldots f_6, \qquad S_2=f_1f_2f_4f_5+f_1f_3f_4f_6+f_2f_3f_5f_6.$$

First, let each $f_i>1$. Since the substitution (12) (45) leaves the triples (4) unaltered, we may set $f_4 \geqq f_5$.

Now $(f_i-1)(f_j-1) \geqq 1$, so that $f_if_j \geqq f_i+f_j$. Thus

$$f_1\ldots f_6 \geqq f_1f_2f_4f_6(f_3+f_5) \geqq f_3f_4f_6(f_1+f_2)+f_1f_2f_5(f_4+f_6).$$

Also $f_3f_4f_6f_2 \geqq f_2f_3f_5f_6$. Hence $f_1\ldots f_6>S_2$, $S_1>S_2$.

Second, let a single f_i be unity. We may set $f_3=1$, $f_5 \geqq f_4$. Then

$$f_1\ldots f_6 \geqq f_1f_2f_5(f_4+f_6) \geqq f_1f_2f_4f_5+f_5f_6(f_1+f_2) \geqq S_2,$$

since $f_1f_5f_6 \geqq f_1f_4f_6$. Hence $S_1>S_2$.

Third, let at least two f_i equal unity. If two such f's belong to the same product in (4_1), we may set $f_2=f_3=1$. Then

$$S_1=f_1+f_4f_5f_6+f_1f_4f_5f_6>f_5f_6+f_1f_4(f_5+f_6), \quad S_1>S_2,$$

if $f_5>1$, $f_6>1$. If $f_5=1$ or $f_6=1$, the triples have the common element 1 and are identical by §2. Next, let the two f's which equal unity belong to different products in (4_1). A case like $f_3=f_6=1$ is excluded by §2. Hence we may set $f_3=1$, $f_4=1$. Thus

$$S_1-S_2=f_2f_5(f_1f_6-f_1-f_6)+f_1f_2+f_5f_6-f_1f_6.$$

If $f_2>1$ or $f_5>1$, $S_1-S_2>(f_2f_5-1)(f_1f_6-f_1-f_6)$, so that $S_1-S_2>0$ unless $f_1=1$ or $f_6=1$. In the latter case the triples (4) have a common element 1. The same is true in the remaining case $f_2=f_5=1$.

4. Theorem. *The volume must contain at least five prime factors.*

By §3, each element exceeds 1. Hence there occur four or more primes. If there are just four primes, the triples without a common element (§2) may be designated

$$(5) \qquad p_1,\ p_2,\ p_3p_4;\ \ p_3,\ p_4,\ p_1p_2,$$

p_4 being the greatest p, and $p_2 \geqq p_1$. Then $S_1+p_1p_2p_3p_4=S_2+p_1p_2p_3p_4$ may be written

$$p_1p_2(p_3-1)(p_4-1)=p_3p_4(p_1-1)(p_2-1).$$

Hence $p_4=p_2$, so that the triples (5) are identical by §2.

5. For five primes, each triple must be of the form (1, 1, 3) or (1, 2, 2), the notation indicating the number of prime factors of each element. Consider first

$$(6) \qquad p_1,\ p_2,\ p_3p_4p_5;\quad p_3,\ p_1p_2,\ p_4p_5,$$

where $p_1 \geqq p_2$. From $S_1+p_1\ldots p_5=S_2+p_1\ldots p_5$, we get

$$p_1p_2(p_3-1)(p_4p_5-1)=p_3p_4p_5(p_1-1)(p_2-1).$$

Hence p_1 divides $p_3p_4p_5$. But $p_1\neq p_3$ by §2. Since (6) is unaltered by the interchange of p_4 and p_5, we may set $p_1=p_4$. Thus

$$(7) \qquad p_2(p_3-1)(p_1p_5-1)=p_3p_5(p_1-1)(p_2-1) \quad (p_1 \geqq p_2).$$

If p_2 does not divide p_1-1, then $p_2=p_5$, and (7) becomes

$$(8) \qquad p_3(p_1+p_2-2)=p_1p_2-1,\ (p_3-1)^2=(p_1-p_3)(p_2-p_3).$$

Let g be the greatest common divisor of $p_1-p_3=ga^2$, $p_2-p_3=gb^2$. Then $p_3-1=gab$,

(9) $p_1=1+gab+ga^2$, $p_2=1+gab+gb^2$, $p_3=1+gab$ (a, b relatively prime)

For these values we have the solution

$$(10) \qquad p_1,\ p_2,\ p_1p_2p_3;\ p_3,\ p_1p_2,\ p_1p_2.$$

For $a=b$, then $a=1$, $p_3=\frac{1}{2}(p_1+1)$ and (10) becomes solution (2). For the remaining cases we may set $a>b$. Examples when the p's are all primes are (2′) and

$a=2$, $b=1$, $g=2$ or 6, p_1, p_2, $p_3=13$, 7, 5 or 37, 19, 13;
$a=3$, $b=2$, $g=10$, $p_1=151$, $p_2=101$, $p_3=61$.

Next, let p_2 divide p_1-1. Then (7) is equivalent to

$$(11) \qquad p_1-1=cp_2,\ p_3-1=kp_5,\ p_1p_5-1=lp_3,\ lk=c(p_2-1).$$

From the second and third we eliminate p_3 and see that $l=-1+mp_5$, $m=p_1-lk$. Hence by the fourth and first, $m=1+c$. Hence (11_3) may be replaced by

(12) $$l=-1+(1+c)p_5.$$

Let g be the greatest common divisor of c and k. Then, by (11_4),

(13) $$c=g\gamma,\quad k=g\mu,\quad l=\lambda\gamma,\quad p_2-1=\lambda\mu \quad (\lambda \text{ and } \mu \text{ relatively prime}).$$

Set $\lambda-gp_5=\rho$. Then (12) becomes $\lambda\rho=p_5-1$. We may thus eliminate p_5 and λ:

(14) $$p_5=1+\gamma\rho,\quad p_2=1+\lambda\mu,\quad p_1=1+g\gamma p_2,\quad p_3=1+g\mu p_5 \quad (\lambda=g+\rho+g\rho\gamma).$$

For any positive integers g, ρ, γ, μ, of which the last two are relatively prime, formulae (14) *give values of the p_i leading to a solution**

(15) $$p_1,\ p_2,\ p_1p_3p_5;\ p_3,\ p_1p_2,\ p_1p_5.$$

For $\gamma=\mu=1$, $p_2=(1+g)(1+\rho)$ is composite. For $\gamma=1$, $\mu=2$, $\rho=1$, the least value of g giving prime p's is $g=10$ and $p_5=2$, $p_2=43$, $p_1=431$, $p_3=41$.

6. For two triples of type (1, 2, 2), it suffices to consider

(16) $$p_1,\ p_2p_3,\ p_4p_5;\ p_2,\ p_1p_4,\ p_3p_5 \quad (p_2>p_1;\ p_5\neq p_1,\ p_2;\ p_3\neq p_4).$$

Then $S_1=S_2$ gives $p_2p_3p_5(p_4-1)\equiv 0 \pmod{p_1}$. If $p_1=p_3$, $S_1=S_2$ gives

$$p_1p_2+p_4p_5+p_2p_4p_5=p_2p_4+p_3p_5+p_1p_4p_5.$$

Thus $p_4p_5(p_1-1)\equiv 0 \pmod{p_2}$, so that $p_2=p_4$. The middle terms in (16) would then be equal. This case is thus excluded by §2. Hence $p_4\equiv 1 \pmod{p_1}$.

Next, $S_1=S_2$ gives $p_1p_4p_5(p_3-1)\equiv 0 \pmod{p_2}$. First, let p_3-1 be prime to p_2, so that $p_4=p_2$. Removing the factor p_2 from $S_1=S_2$, we get

$$p_1(p_2-p_3-p_5)=p_3p_5(p_2-p_1-1).$$

Since $p_2\equiv 1 \pmod{p_1}$, $p_2-p_1-1=cp_1$, where c is an integer $\geqq 0$. Thus

(17) $$p_4=p_2=(c+1)p_1+1=p_3+p_5+cp_3p_5.$$

For $c=0$, we set $p_1=p$, $p_3=q$ and obtain solution (3); note that p occurs in the second triple if and only if $q=1$ or p.

*To show that these triples are distinct, it suffices to prove that p_1 does not occur in the second. But $p_1<p_1p_2$, $p_1<p_1p_5$. If $p_1=p_3$, then $\gamma p_2=\mu p_5$. But γ and μ are relatively prime, and γ does not divide p_5 by (14_1).

For $c=1$, $2(p_1+1)=(p_3+1)(p_5+1)$. If $p_3=2$, then $p_1=3k-1$, $p_5=2k-1$. The lowest values of k leading to prime values for each p_i are $k=2, 4, 10$:

$$p_1,\ p_5,\ p_2=5,\ 3,\ 11;\ 11,\ 7,\ 23;\ 29,\ 19,\ 59.$$

The first gives the solution 5, 22, 33; 6, 11, 55. If p_3 is odd, then

$$p_3=2k-1,\quad p_1=k(p_5+1)-1,\quad p_4=p_2=2k(p_5+1)-1. \tag{18}$$

The p_i are all primes for $k=2$, $p_5=5$, 11; $k=3$, $p_5=7$, 13:

$$p_5,\ p_3,\ p_1,\ p_2=5,\ 3,\ 11,\ 23;\ 11,\ 3,\ 23,\ 47;\ 7,\ 5,\ 23,\ 47;\ 13,\ 3,\ 41,\ 83.$$

For $c=2$, we have $p_3=3$, $p_5=3k+1$, $p_1=7k+3$, or

$$p_3=3l+1,\quad p_1=l+p_5+2lp_5,\quad \text{or}\quad p_3=3l+2,\quad p_5=3m+1,\quad p_1=6lm+3l+5m+2.$$

But for c even, the p's in (17) are not all primes if $p_1\neq 2$.

In view of the variety of solutions obtained, further cases are not considered.

7. Do there exist three distinct cuboids with rational edges having equal volumes and equal surfaces? We show that there is no solution in which the volume is the product of fewer than six primes. In view of §4, we consider the case of three triples of five primes p_i. Not all three are of the type (1, 1, 3). Let two be of type (1, 1, 3) and one of type (1, 2, 2); it suffices to consider

$$p_1,\ p_2,\ p_3p_4p_5;\ p_3,\ p_4,\ p_1p_2p_5;\ p_5,\ p_1p_2,\ p_3p_4,\ \text{or}\ p_5,\ p_1p_3,\ p_2p_4. \tag{19}$$

In the first alternative, we reduce the S_i modulo p_1 and get

$$p_2p_3p_4p_5\equiv p_3p_4\equiv p_3p_4p_5 \pmod{p_1}.$$

Since p_3p_5 is not divisible by p_1, $p_5\equiv 1$, $p_2\equiv 1 \pmod{p_1}$. Similarly $p_1\equiv 1 \pmod{p_2}$, contrary to the former. Likewise (19_2) gives $p_3\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_3}$ and is excluded.

Next, let one triple be of the type (1, 1, 3) and two of the type (1, 2, 2); it suffices to treat

(20) $(p_1,\ p_2p_3,\ p_4p_5;\ p_2,\ p_1p_4,\ p_3p_5;\ p_3,\ p_4,\ p_1p_2p_5,$ or $p_4,\ p_5,\ p_1p_2p_3.$

But, in either case, we find that $p_4\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_4}$.

Hence all three triples must be of type (1, 2, 2). Two of them may be taken to be the first two in (20), with $p_3\neq p_4$, $p_2\neq p_5$, $p_1\neq p_5$. Since these are unaltered by (12) (34), the third may be restricted to one of the three:

$$p_3,\ p_1p_5,\ p_2p_4;\ p_5,\ p_1p_2,\ p_3p_4;\ p_5,\ p_1p_3,\ p_2p_4.$$

The first case gives $p_3\equiv 1 \pmod{p_2}$, $p_2\equiv 1 \pmod{p_3}$. The second case gives $p_2\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_2}$. In the third case we may set $p_2>p_1$; then

$$p_1p_4p_5\equiv p_1p_2p_4\equiv p_2p_4p_5 \pmod{p_3},\quad p_1p_4p_5\equiv p_1p_3p_4p_5\equiv p_1p_3p_5 \pmod{p_2}.$$

If $p_3=p_5$, then $p_1p_2p_4$ is a multiple of p_3, whereas $p_1\neq p_5$, $p_2\neq p_5$, $p_4\neq p_3$. Hence $p_4p_5(p_2-p_1)\equiv 0$, gives $p_2-p_1=cp_3$, $c>0$. Similarly, $p_1p_4p_5(p_3-1)\equiv 0 \pmod{p_2}$ gives $p_3\equiv 1 \pmod{p_2}$, since $p_4=p_2$ would require $p_1p_3p_5\equiv 0 \pmod{p_2}$. Hence $-p_1\equiv c \pmod{p_2}$, so that $c\geqq p_2-p_1$. The former equation $p_2-p_1=cp_3$ is thus impossible in view of $p_3>1$. *There is no set of three triples involving fewer than six primes.*

8. Consider three triples involving six primes. If all are of type (1, 1, 4), they may be taken to be

(21) $p_1,\ p_2,\ p_3p_4p_5p_6;\ p_3,\ p_4,\ p_1p_2p_5p_6;\ p_5,\ p_6,\ p_1p_2p_3p_4,$

with $p_3p_4>p_5p_6$. By $S_2=S_3$, $p_3p_4=p_5p_6+\delta p_1p_2$, $\delta>0$. But $p_1p_2\equiv p_5p_6 \pmod{p_3p_4}$ by $S_1=S_3$. Hence $0\equiv p_5p_6(1+\delta) \pmod{p_3p_4}$. Thus $\delta=-1+\epsilon p_3p_4$, $\epsilon>0$,

$$p_3p_4+p_1p_2=p_5p_6+\epsilon p_1p_2p_3p_4.$$

But $p_1p_3.p_2p_4\geqq (p_1+p_3)(p_2+p_4)>p_1p_2+p_3p_4$. Hence this case is excluded.

Let two of the triples be the first two in (21) and the third of type (1, 2, 3). Since the former are unaltered or interchanged by (12), (34), (56), (13) (24), the third may be assumed to be $p_5,\ p_1p_2,\ p_3p_4p_6$; or $p_5,\ p_1p_3,\ p_2p_4p_6$; or $p_5,\ p_1p_6,\ p_2p_3p_4$. The first yields $p_2\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_2}$; the second, $p_3\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_3}$. For the third case, $S_1=S_2$ gives

(22) $\quad p_1p_2=p_3p_4+\delta p_5p_6, \quad \delta+p_1p_3p_4+p_2p_3p_4=p_1p_2p_3+p_1p_2p_4,$

while $S_1=S_3$ gives $p_5p_6\equiv p_2 \pmod{p_3p_4}$. Thus $\delta\equiv p_1 \pmod{p_3p_4}$, by (22_1). Then (22_2) gives $p_1\equiv p_1p_2(p_3+p_4)$, mod p_3p_4. Hence

(23) $$1=p_2(p_3+p_4)+lp_3p_4 \quad (l \text{ negative integer}).$$

But $p_1p_3p_4p_5p_6\equiv p_3p_4\equiv p_1p_5p_6 \pmod{p_2}$. Thus $p_1p_5p_6\equiv 1$, $p_3p_4\equiv 1 \pmod{p_2}$. Hence in (23), $1\equiv l$, $l=1-mp_2$, $m>0$. Thus (23) may be given the form

$$p_2[(m-1)p_3p_4-p_3-p_4]+p_2p_3p_4=p_3p_4-1.$$

But if $m>1$, the quantity in brackets is positive and the equation impossible. Hence $m=1$, and (23) becomes

(24) $$p_2(p_3p_4-p_3-p_4)=p_3p_4-1.$$

Since $p_2\geqq 2$, the first member exceeds the second if $(p_3-2)(p_4-2)>3$. Since (34) leaves each triple unaltered, we may set $p_3\geqq p_4$. Hence, there remains the cases

$$p_3=5,\ p_4=3;\ p_3=p_4=3;\ p_4=2.$$

For $p_3=5$, $p_4=3$, we have $p_2=2$, $S_2=15+16p_1p_5p_6$, $S_3=30p_5+30p_1p_6+p_1p_5p_6$. Hence $S_2=S_3$ gives $(p_1p_6-2)(p_5-2)=3$, $p_1p_6=5$ or 3, which is impossible. For $p_3=p_4=3$, (24) gives $3p_2=8$. Hence must $p_4=2$, $p_2(p_3-2)=2p_3-1$. Thus $p_2\geqq 3$, and the first member exceeds the second if $p_3>5$. Hence $p_3=5$, $p_2=3$, or $p_3=3$, $p_2=5$. For the first, $2S_2=2S_3$ becomes $(2p_5-3)(2p_1p_6-3)=7$, whence $p_1p_6=5$ or 2. For the second, $4S_2=4S_3$ becomes $(4p_1p_6-5)(4p_5-5)=21$, whence $p_1p_6=3$ or 2.

Next, let two of the triples be the first two in (21) and the third of type (2, 2, 2). The latter may be assumed to be p_1p_2, p_3p_4, p_5p_6; p_1p_2, p_3p_5, p_4p_6; p_1p_3, p_2p_4, p_5p_6; or p_1p_3, p_2p_5, p_4p_6. The first two and the last two are excluded by the argument excluding the first and second cases, respectively, at the beginning of the preceding paragraph.

Hence at most one of the three triples is of the type (1, 1, 4).

9. Let two triples be p_1, p_2, $p_3p_4p_5p_6$ and p_3, p_1p_2, $p_4p_5p_6$. Adding $p_1\ldots p_6$ to S_1 and S_2 and equating the sums, we get

(25) $$p_1p_2(p_3-1)(p_4p_5p_6-1)=p_3p_4p_5p_6(p_1-1)(p_2-1).$$

We may set $p_2 \geqq p_1$, $p_2 \neq p_3$. Hence p_2 divides $p_4p_5p_6$, so that we may set $p_4=p_2$. Since $p_4p_5p_6-1>p_5p_6(p_2-1)$, we have $p_1(p_3-1)<p_3(p_1-1)$, $p_1>p_3$. Hence $p_2>p_3$. If the third triple is of type (1, 2, 3), it may be taken to be p_5, p_1p_3, $p_2^2p_6$; p_5, p_1p_6, $p_2^2p_3$; p_5, p_2p_3, $p_1p_2p_6$; p_5, p_2^2, $p_1p_3p_6$; p_5, p_2p_6, $p_1p_2p_3$; or p_5, p_3p_6, $p_1p_2^2$. Now S_1 is a multiple of p_2, so that S_3 must be. Hence for the first case, $p_1p_3p_5$ is a multiple of p_2, whence $p_1=p_2$. Then S_1 is a multiple of p_2^2, so that S_3 must be; hence $p_3p_5\equiv 0 \pmod{p_2}$, which is impossible. For the second case, $p_1p_5p_6\equiv 0$ $\pmod{p_2}$. If $p_1=p_2$, $S_1\equiv 0 \pmod{p_2^2}$, so that by S_3, $p_6=p_2$. In any event, $p_6=p_2$ and the second and third triples have $p_1p_2=p_1p_6$ in common. For the third case, $S_1=S_2$ gives $p_1p_3=p_1+\varepsilon p_5p_6$, whence $\varepsilon=\delta p_1$; while $S_1=S_3$ gives $p_1p_2\equiv p_1p_2p_5p_6$, $p_5p_6\equiv 1 \pmod{p_3}$, $p_5p_6>p_3$. For the fourth case, $S_1\equiv S_3$ mod p_2 gives $p_1p_3p_6p_5\equiv 0 \pmod{p_2}$. If $p_1=p_2$ the second and third triples would have a common element. Hence $p_2=p_6$. By (25), p_5p_6 must divide $p_1(p_3-1)$. But $p_2>p_3-1$. Hence $p_2=p_6$ requires $p_1=p_2$. For the fifth case, $p_1\neq p_6$ by p_2p_6 and p_1p_2. Removing the factor p_2 from the S's, we see that $p_2p_3p_5p_6\equiv p_3p_5p_6\equiv p_5p_6 \pmod{p_1}$, $p_3\equiv 1 \pmod{p_1}$, contrary to $p_3<p_1$. For the sixth case, $S_1\equiv S_3 \pmod{p_2}$ gives $p_3p_5p_6\equiv 0$ $\pmod{p_2}$, $p_6=p_2$. As in the fourth case, $p_1=p_2$. Then S_1, but not S_3, is a multiple of p_2^2.

If the third triple is of type (2, 2, 2), it may be taken to be

$$p_2^2,\ p_1p_3,\ p_5p_6;\ p_2^2,\ p_1p_5,\ p_3p_6;\ p_2p_3,\ p_2p_5,\ p_1p_6;\ \text{or}\ p_2p_5,\ p_2p_6,\ p_1p_3.$$

For the first two cases, $p_1\neq p_2$ and $S_1\equiv S_3 \pmod{p_2}$ gives $p_6=p_2$, contrary to the above. The third case is excluded by $p_3\equiv 1 \pmod{p_5p_6}$, $p_5p_6\equiv 1$ $\pmod{p_3}$; the fourth by $p_1\equiv 1 \pmod{p_3}$, $p_3\equiv 1 \pmod{p_1}$.

THE UNIVERSITY OF CHICAGO, *May 16, 1909.*

NOTE ON SOME POLYNOMIALS RELATED TO LEGENDRE'S COEFFICIENTS.*

By R. D. CARMICHAEL, Anniston, Alabama.

The object of this note is to point out some properties of a class of functions which contains Legendre's coefficients as a special case. It will be seen that the former possess some interesting properties belonging to the latter.

1. Consider the definite integral†

*Presented to the American Mathematical Society, April 24, 1909.
†Cf. Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 173.

$$\int_a^b Q\frac{d^n R}{d\,x^n}dx=0,$$

where Q is any polynomial of degree less than n. It is required to find a polynomial d^nR/dx^n of degree $n(2\alpha-1)$, α odd, satisfying the above equation. Since we desire only the value of d^nR/dx^n we may choose n zeros of R at will. Let each of them be a, so that $(x-a)^n$ is a factor of R. Now, integrating by parts, we have

$$\int_a^b Q\frac{d^n R}{dx^n}d\,x\Big[QR^{(n-1)}-Q'R^{(n-2)}+Q''R^{(n-3)}-\ldots\pm Q^{(n-1)}R\Big]_a^b,$$

where the indices in the brackets indicate differentiation with respect to x. Since $(x-a)^n$ is a factor of R,

$$R(a)=0,\ R'(a)=0,\ R''(a)=0,\ \ldots,\ R^{(n-1)}(a)=0;$$

and therefore,

$$Q(b)R^{(n-1)}(b)-Q'(b)R^{(n-2)}(b)+\ldots\pm Q^{(n-1)}(b)R(b)=0;$$

or, since Q is arbitrary by hypothesis,

$$R^{(n-1)}(b)=0,\ R^{(n-2)}(b)=0,\ \ldots,\ R(b)=0.$$

Hence $(x-b)^n$ is a factor of R; that is,

$$R=c(x-a)^n(x-b)^nf(x),$$

where $f(x)$ is an arbitrary polynomial of degree $2n(\alpha-1)$ and c is a constant. Hence,

(1) $$\int_a^b Q\frac{d^n}{dx^n}[c(x-a)^n(x-b)^nf(x)]\,dx=0,$$

where Q is any polynomial of degree less than n and $f(x)$ is any polynomial of degree $2n(\alpha-1)$.

Since $f(x)$ is of degree $2n(\alpha-1)$, α odd,* it is evident that we may choose $f(x)$ so that

$$R=c(x^\alpha-a^\alpha)^n(x^\alpha-b^\alpha)^n,$$

and therefore

(2) $$\int_a^b Q\frac{d^n}{dx^n}[c(x^\alpha-a^\alpha)^n(x^\alpha-b^\alpha)^n]\,dx=0.$$

*This is the first use of the fact that α is odd.

Now take $a=-1$, $b=1$, $c=\frac{1}{2^n.\ n!}$, and set

$$P_{n\alpha}=\frac{1}{2^n.\ n!}\frac{d^n}{dx^n}(x^{2\alpha}-1)^n,$$

so that $P_{n1}=P_n$ in the ordinary notation for Legendre's coefficients. Then by (2),

$$\int_{-1}^{+1} QP_{n\alpha}dx=0,$$

where Q is any polynomial of degree less than n. In particular,

$$\int_{-1}^{+1} P_{m1}P_{n\alpha}dx=0,\ m<n,$$

a generalization of the formula $\int_{-1}^{+1} P_mP_ndx=0,\ m\neq n.$

If we write $P_{n\alpha}$ in the form

$$P_{n\alpha}=\frac{1}{2^n.\ n!}\ \frac{d^n}{dx^n}(x^\alpha-1)^n(x^\alpha+1)^n,$$

from the indicated nth derivative of the product by Leibniz' formula, and in the result give to x the values $+2$ and -1, we have

$$P_{n\alpha}(1)=\alpha^n,\ P_{n\alpha}(-1)=(-\alpha)^n,$$

a generalization of the corresponding formulae for Legendre's coefficients.

Again, since two real roots of $\frac{d^s}{dx^s}(x^\alpha-1)^n(x^\alpha+1)^n$, $s<n$, are -1 and $+1$, it follows from Rolle's Theorem that $P_{n\alpha}$ has n real roots between -1 and $+1$.

2. If x, y, z are connected by the relation

$$z=x+y\phi(z),$$

then by Lagrange's formula we have

$$f(z)=f(x)+y\phi(x)f'(x)+\frac{y^2}{2!}\frac{d}{dx}[\phi(x^2)f'(x)]+\ldots$$

$$+\frac{y^{n-1}}{(n-1)\,!}\frac{d^{n-2}}{dx^{n-2}}[\phi(x)^{n-1}f'(x)]+\ldots$$

Therefore if we assume the special relation

$$z=x+\frac{y}{2}(z^{2a}-1),$$

we may write the value of z in the form

$$z=x+\frac{y}{2}(x^{2a}-1)+\frac{1}{2\,!}\left(\frac{y}{2}\right)^2\frac{d}{dx}(x^{2a}-1)^2+\frac{1}{3\,!}\left(\frac{y}{2}\right)^3\frac{d^2}{dx^2}(x^{2a}-1)^3+\ldots$$
$$+\frac{1}{n\,!}\left(\frac{y}{2}\right)^n\frac{d^{n-1}}{dx^{n-1}}(x^{2a}-1)^n+\ldots$$

Differentiating with respect to x, regarding y as constant, we have the formula

$$\frac{dz}{dx}=1+\frac{y}{2}\frac{d}{d\,x}(x^{2a}-1)+\frac{1}{2\,!}\left(\frac{y}{2}\right)^2\frac{d^2}{dx^2}(x^{2a}-1)^2+\ldots$$
$$+\frac{1}{n\,!}\left(\frac{y}{2}\right)^n\frac{d^n}{dx^n}(x^{2a}-1)^n+\ldots;$$

or, defining $P_{0a}=1$, this expansion may be written,

$$\frac{dz}{dx}=P_{0a}+yP_{1a}+y^2P_{2a}=y^3P_{3a}+\ldots$$

an equation which holds subject to the sole condition that x, y, z are connected by the relation (3). If $a=1$, dz/dx becomes $(1-2xy+y^2)^{-\frac{1}{2}}$ as may be easily shown by solving (2) for z and differentiating the result with respect to x; and therefore we have the well-known expansion of $(1-2xy+y^2)^{-\frac{1}{2}}$ in ascending powers of y, the coefficients being Legendre's polynomials.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

313. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the equations $px^2+qx+r=0$ and $\alpha x^2+\beta x+\gamma+y(ax^2+bx+c)=0$ may give equal values for y.

Solution by G. W. HARTWELL, University of Kansas, Lawrence, Kansas.

$$px^2+qx+r=0\ldots(1).$$
$$\alpha x^2+\beta x+\gamma+y(ax^2+bx+c)=0\ldots(2).$$

Since (2) is of the first degree in y, the values of y will be equal when the roots of (1) are equal.

$\therefore q^2-4pr=0$ is one condition that (1) and (2) give equal values of y.

If $q^2-4pr\neq 0$, we have on solving (1) and substituting in (2),

$$y=-\frac{\alpha q^2-2\alpha pr-\beta pq+2\gamma p^2\pm(\beta p-\alpha q)\sqrt{(q^2-4pr)}}{aq^2-2apr-bpq+2cp^2\pm(bp-aq)\sqrt{(q^2-4pr)}}\ldots(3).$$

The two values given by (3) will be equal when,

$$(bp-aq)(\alpha q^2-2\alpha pr-\beta pq+2\gamma p^2)$$
$$-(\beta p-\alpha q)(aq^2-2apr-bpq+2cp^2)=0\ldots(4).$$

(4) can be put in the form:

$$p^2\begin{vmatrix} p & q & r \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix}=0.$$

$\therefore$ The conditions that (1) and (2) give equal values of y are

I. $q^2-4pr=0$. Or, II. $\begin{vmatrix} p, & q, & r \\ a, & b, & c \\ \alpha, & \beta, & \gamma \end{vmatrix}=0$. Or, III. $p=0$.

In this case (1) becomes an equation of the first degree, and hence (2) can give only one value of y.

Also solved by V. M. Spunar.

314. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Sum to infinity the series $\frac{1}{2.3.3.4}+\frac{1}{4.5.5.6}+\frac{1}{6.7.7.8}+\frac{1}{8.9.9.10}+\ldots$

I. Solution by E. B. ESCOTT, Ann Arbor, Mich., and G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The general term of the series is $\frac{1}{2n(2n+1)^2(2n+2)}$, which may be resolved into the partial fractions,

$$\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1.}{(2n+1)^2}\Big).$$

Therefore,

$$\frac{1}{2.3.3.5}=\frac{1}{4}\left(1-\frac{1}{2}\right)-\frac{1}{3^2}$$

$$\frac{1}{4.5.5.6}=\frac{1}{4}\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{5^2}$$

$$\frac{1}{6.7.7.8}=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{4}\right)-\frac{1}{7^2}$$

. . . . . .

Adding, we have

$$\frac{1}{2.3.3.4}+\frac{1}{4.5.5.6}+\ldots=\frac{1}{4}-\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots.\right)=\frac{1}{4}-\left(\frac{\pi^2}{8}-1\right).$$

[See E. Pascal, *Repertorium der hoeheren Mathematik*, Vol. 1, p. 60.]

The sum is, therefore, $\frac{5}{4}-\pi^2/8=.0163$.

II. Solution by S. A. COREY, Hiteman, Iowa.

Take the Fourier's cosine series for $x\sin x$, viz.,

$$x\sin x=1-\frac{\cos x}{2}-\frac{2\cos 2x}{1.3}+\frac{2\cos 3x}{2.4}-\frac{2\cos 4x}{3.5}+\ldots.$$

and integrate both members of the equation three times to obtain the only constant of integration involved, viz., $(\frac{1}{2}+\pi^2/12)$, in the series obtained by the second integration. The latter series may then be thus written,

$$\frac{-x\sin x}{2}-\frac{5\cos x}{4}-\frac{x^2}{4}+\frac{1}{2}+\frac{\pi^2}{12}=\frac{\cos 2x}{1.2.2.3}-\frac{\cos 3x}{2.3.3.4}+\frac{\cos 4x}{3.4.4.5}-\ldots(1).$$

When $x=0$, (1) becomes

$$\frac{1}{1.2.2.3}-\frac{1}{2.3.3.4}+\frac{1}{3.4.4.5}-\frac{1}{4.5.5.6}+\ldots=\frac{\pi^2}{12}-\frac{3}{4}\ldots(3).$$

When $x=\pi$, (1) becomes

$$\frac{1}{1.2.2.3}+\frac{1}{2.3.3.4}+\frac{1}{3.4.4.5}+\frac{1}{4.5.5.6}+\ldots=\frac{7}{4}-\frac{\pi^2}{6}\ldots(4).$$

Subtracting (3) from (4), we find the sum of the given series to be $\frac{5}{4}-\pi^2/8$.

Also solved by J. Scheffer.

315. Proposed by PROFESSOR B. F. YANNEY, Mount Union College, Alliance, Ohio.

Simplify, $1-(2-(3-\ldots-(n-1)-n)\ldots)))$.

Solution by GEORGE W. HARTWELL, University of Kansas, Lawrence, Kansas, and V. M. SPUNAR, Pittsburg, Pa.

Removing the parentheses, this expression becomes

$$1-2+3-4+\ldots(-1)^{n-1}n\equiv\sum_1^n(-1)^{n-1}n.$$

But $\sum_1^n(-1)^{n-1}n=-(n/2)$ when n is even,

and $\sum_1^n(-1)^{n-1}n=(n+1)/2$ when n is odd.

Also solved by G. B. M. Zerr.

GEOMETRY.

342. Proposed by G. I. HOPKINS, M. A., Instructor in Mathematics and Astronomy, Manchester, N. H.

Given, circle DEF inscribed in triangle ABC and circumscribing the triangle DEF, D, E, F being the points of contact; AH is drawn through center, N, meeting chord DF in H. Through H is drawn BK meeting AC in K. Prove triangle ABK isosceles.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let the points D, F, E be situated on the sides a, b, c, respectively, and also let $l=\cos^2(A/2)$, $m=\cos^2(B/2)$, $n=\cos^2(C/2)$. Then $(0, rn, rm)$; $(rn, 0, rl)$, are the trilinear coordinates of D and F, respectively.

Hence $\beta-\gamma=0$ is the equation to AN, $l\alpha+m\beta-n\gamma=0$ is the equation to DF.

$\therefore (n-m,\ l,\ l)$; $(0,\ 2\triangle/b,\ 0)$, are the coordinates of H and B, respectively. $\therefore l\alpha+(m-n)\gamma=0$ is the equation to BK.

But $m-n+l\cos C+(m-n)\cos A-l\cos B=0$.

$\therefore BK$ is perpendicular to AN. $\therefore$ triangle ABK is isosceles.

II. **Solution by G. I. HOPKINS, Instructor in Mathematics and Astronomy, High School, Manchester, N. H.**

Construction: Join HE, HB. Extend DE and draw BP perpendicular to it.

Demonstration: Since BP and AH are perpendicular to DE, they are parallel. $AD=AE$, *i. e.*, $\triangle ADE$ is isosceles. arc ES=arc SF.

$\therefore \angle ENB=\angle EDF$. $\therefore \triangle DRH$ and $\triangle NEB$ are similar.

$\therefore NE/EB=DR/RH$. $\angle NEB$ is a right angle.

$\therefore \angle REN$ is complement to $\angle BEP$. $\therefore \angle REN=\angle EBP$.

$\therefore \triangle REN$ and $\triangle EBP$ are similar.

$\therefore RE/BP=NE/EB$; $\therefore DR/RH=RE/BP$.

Since $DR=RE$, $\therefore RH=BP$, and $\therefore RHBP$ is a parallelogram; *i, e.*, BK is parallel to DE.

$\therefore \triangle ABK$ is similar to $\triangle ADE$, and is therefore isosceles.

CALCULUS.

270. **Proposed by S. A. COREY, Hiteman, Iowa.**

Prove that $\sum_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}=\frac{\pi}{2a^{2n-1}}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}+\frac{1}{2a^{2n}}$, n being a positive integer >1.

II. **Solution by the PROPOSER.**

Performing the finite summation of the problem by the aid of Maclaurin's Summation-formula,

$$\Sigma\, u_x=C+\int u_x dx-\tfrac{1}{2}u_x+\frac{B_1}{2!}\frac{du_x}{dx}-\frac{B_2}{4!}\frac{d^3u_x}{dx^3}+\dots$$

(See Boole's *Finite Differences*, page 90), we readily obtain the above expression for the sum, if we substitute for the definite integral, $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$, its well known value, $\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}\cdot\frac{1}{a^{2n-1}}$, if n is a positive integer >1.

The solution in the May MONTHLY involves the erroneous assumption that $\sum_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}=\int_0^\infty\frac{dx}{(a^2+x^2)^n}$.

271. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-5\left(\frac{d^2y}{dx^2}\right)\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\tfrac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

show that there is an integrating factor of the form $\left(\frac{d^2y}{dx^2}\right)^n$, and integrate the equation.

Solution by the PROPOSER, and LEVI S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

When the equation is multiplied by $(d^2y/dx^2)^n$, if it is an exact equation, its first integral is

$$\left(\frac{d^2y}{dx^2}\right)^{n+2}\frac{d^4y}{dx^4}+\frac{40}{9(n+1)}\left(\frac{d^2y}{dx^2}\right)^{n+1}\left(\frac{d^3y}{dx^3}\right)^2=c_1 \ldots (2).$$

Differentiating this equation and comparing with original equation, we find that $n=-\frac{11}{3}$ and $-\frac{13}{3}$.

In (2), put $n==\frac{11}{3}$. The equation is exact and its integral is

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{5}{3}}\frac{d^3y}{dx^3}=c_1x+c_2.$$

Integrating again,

$$-\tfrac{3}{2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=\frac{c_1}{2}x^2+c_2x+c_3.$$

Solving for d^2y/dx^2, and changing the constants,

$$\frac{d^2y}{dx^2}=\frac{1}{(c_1x^2+2c_2x+c_3)^{\frac{3}{2}}}.$$

Integrating twice, we have

$$y=\frac{1}{c_1c_3-c_2^2}\sqrt{(c_1x^2+2c_2x+c_3)}+c_4x+c_5.$$

Also solved by G. B. M. Zerr, and V. M. Spunar.

272. Proposed by CLARENCE OHLENDORF, Chicago, Ill.

Find $\int \log_e \tan^{-1} x dx$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and C. N. SCHMALL, New York City.

Putting $x=\tan y$, we have

$$\int \log \tan^{-1}x dx=\int \log y\frac{dy}{\cos^2 y}=\int \log y \sec^2 y dy=\log y-\tan y-\int\frac{\tan y dy}{y}.$$

Substituting for $\tan y$ its series, and integrating, we get

$$\int\frac{\tan y dy}{y}=\frac{2^2(2^2-1)}{2!}B_1 y+\frac{2^4(2^4-1)}{4!}B_3\frac{y^3}{3}+\frac{2^6(2^6-1)}{6!}B_5\frac{y^5}{5},$$

where B_1, B_3, B_5, ... are Bernoulli's numbers.

Therefore the given integral is

$$x\log\tan^{-1}x-\left[\frac{2^2(2^2-1)}{2!}B_1\tan^{-1}x+\frac{2^4(2^4-1)}{4!}B_3\frac{(\tan^{-1}x)^3}{3}+\ldots\right].$$

Also solved by G. B. M. Zerr, V. M. Spunar, and J. E. Sanders.

273. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

On one side of a circular pond a feet in radius is a duck. On the diametrically opposite side of the pond is a dog. Both begin to swim at the same time, the duck swimming around the circumference of the pond at the rate of m feet a minute, the dog swimming directly towards the duck at the rate of n feet per minute. How far will the dog swim in overtaking the duck?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

If the line joining the positions of the dog and the duck is tangent to the dog's path, the following is the solution.

Let A, B be the starting points of dog and duck; O, the center of the pond; P, R, corresponding positions of the dog and duck. Then the tangent t, from P must pass through R. Let the duck swim to S and the dog to Q. Draw OF, PH perpendicular to AB, PI, QD perpendicular to OF, QC perpendicular to PI, and RE perpendicular to QS, produced.

Let $AB=2a$, $AP=\sigma$, $\angle RAB=\phi$, $OD=x$, $n/m=b$. Then $PR=d\sigma$, $\angle RAS=d\phi$, $DI=QC=dx$.

Now $\sigma=2ab\phi$; $\therefore d\sigma/2ad\phi=b$.

In the limit the triangles PQC and RES are similar.

$\therefore d\sigma : dx=2ad\phi : ES$. $\therefore ES=(2ad\phi/d\sigma)dx=dx/b$.

The tangent t has negative increments at both ends (PQ and ES).

$\therefore dt=-d\sigma-dx/b$, or $t=C-\sigma-x/b$. When $t=2a$, $\sigma=0$, $x=0$.

$\therefore C=2a$. $\therefore t=2a-2ab\phi-x/b$. When $t=0$, $x=2a\sin\phi\cos\phi$.

$\therefore b=b^2\phi+\sin\phi\cos\phi=(b^2+1)\phi-\frac{2}{3}\phi^3+\frac{2}{15}\phi^5-\frac{4}{315}\phi^7+\ldots$

$$Q=\frac{b}{b^2+1}+\frac{2b^4}{3(b^2+1)^4}+\frac{2b^5(9-b^2)}{15(b^2+1)^7}+\frac{4(225+b^4-54b^2)b^7}{315(b^2+1)^{10}}+\ldots$$

by reversion of series.

b. If the dog always keeps on the line joining his starting point to the duck's position, the solution becomes quite simple.

Then $us=2a\psi$ is the intrinsic equation to the dog's path.

$$\therefore \frac{ds}{d\psi}=\frac{2a}{u}=\sqrt{r^2+\left(\frac{dr}{d\psi}\right)^2}. \therefore d\psi=\frac{dr}{\sqrt{\frac{4a^2}{u^2}-r^2}}, \text{ and } \psi=\sin^{-1}\frac{ur}{2a}.$$

When $r=2a\cos\psi$ the dog catches the duck.

$\therefore \psi=\sin^{-1}(u\cos\psi)$, or $\tan\psi=u$, and $\psi=\tan^{-1}u$.

$\therefore us=2a\tan^{-1}u$, or $s=(2a/u)\tan^{-1}u=(2an/m)\tan^{-1}(m/n)$ is the distance the dog swims to overtake the duck.

MECHANICS.

225. Proposed by W. A. BALDWIN, Springfield, Mo.

Find, *by means of polar coordinates*, the moment of inertia about the origin of the area between the parabola $ay=2(a^2-x^2)$, the circle $x^2+y^2=a^2$, and the axis of Y.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The polar equation to the parabola is

$$r=\frac{a[\sqrt{(1+15\cos^2\theta)}-\sin\theta]}{4\cos^2\theta}=r_1.$$

The polar equation to the circle is $r=a$.

$\therefore m$, moment of inertia required,

$$=\int_{\pi/6}^{\pi/2}\int_a^{r_1} r^3 d\theta\, dr=\frac{a^4}{1024}\int_{\pi/6}^{\pi/2}[8\sec^8\theta+112\sec^6\theta+136\sec^4\theta-256$$

$$-8\sin\theta\sec^6\theta(7+\sec^2\theta)\sqrt{(1+15\cos^2\theta}]d\theta=\frac{a^4}{4}\left(\frac{883\sqrt{3}}{280}-\frac{\pi}{3}\right).$$

This is problem 7, p. 350, Osborne's *Differential and Integral Calculus*. Osborne gives as the result, $(\frac{52}{35}-\frac{1}{8}\pi)a^4$. Professor Zerr's result is correct. ED. F.

strokes cursively, the two was two horizontal parallel bars and the eight two curved vertical lines, and some of the old forms of eight are practically the same as the two turned over 90°.

It is quite likely also that when forms of numerals are evidently tally marks, the ancient tribes would not stick to any particular arrangement, but form new ones provided they indicated numbers. This is the most reasonable explanation of the very evident tally-mark nature of the numerals in the Jaina manuscript. Ten is a nine with an extra stroke, and the eights are sevens with an extra stroke. The Jaina four, five, and six are also clearly derived from groups of marks. In course of time, by slurring, omission of strokes and adding embellishing flourishes, the Nepal and Bower manuscript forms arose. Indeed in the seven there is a perfect gradation of evolutionary forms to our present seven. In the four the resemblance is seen by making an assumption. In the five there is more evidence of an attempt to write cursively one of the X forms of the Chinese, but the six is not so evident without making two assumptions.

The supremacy of the Chinese numerals is explained by the fact that they were the first ideographs in the field. Egyptian pictographs evolved in the direction of representing sounds and, besides, their tally marks elsewhere were in groups of parallels, and not the fortunate Chinese groupings which lent themselves to change into ideographs. The invention of position value of course killed all the numerals above nine.

ON THE NUMBER OF EQUAL REGULAR SPHERICAL POLYGONS THAT CAN BE CONSTRUCTED SO AS TO COMPLETELY COVER A SPHERE.

By B. F. YANNEY, Alliance, Ohio.

Let N=the number of equal polygons required; n=the number of sides in each, and k=the number of angles about each common vertex.

Then will $[\frac{360}{k}n-(n-2)180]N$=the area, in spherical degrees, of the sum of all the spherical polygons completely covering the sphere; but this area is also equal to 720.

Therefore, $[\frac{360}{k}n-(n-2)180]N=720$; whence, $N=\frac{4k}{2n-nk+2k}$.*

It remains to solve this equation for positive integers.

*This formula may be found on page 69, Vol. III, of Henrici and Treutline's *Geometry*, though developed by a different method, and having different considerations in view.

1. For $k=1$, $N=\frac{4}{n+2}$. Now, the only positive integral value n can have, to make N likewise integral, is 2; from which $N=1$. This is the case of a lune with its sides coincident and its angle 360°.

2. For $k=2$, $N=2$ for any value of n. This is the case of two hemispheres, each of which may be considered as bounded by any number of sides, each angle being an angle of 180°. In particular, n may equal 2, in which case each hemisphere is regarded as a lune.

3. For $k=3$, $N=\frac{12}{6-n}=3$, 4, 6, or 12, according as $n=2$, 3, 4, or 5, respectively. The figures are, in order, lunes, equilateral triangles, regular quadrangles, and regular pentagons.

4. For $k=4$, $N=\frac{8}{4-n}=4$ or 8, according as $n=2$ or 3. The corresponding figures are lunes or equilateral triangles.

5. For $k=5$, $N=\frac{20}{10-3n}=5$ or 20, according as $n=2$ or 3. The corresponding figures are lunes or equilateral triangles.

6. For $k=6$, $N=\frac{24}{12-4n}=6$, when $n=2$. [Lunes.] When $n=1$, $N=3$. But $k=6$, $n=1$, $N=3$ are incompatible with the nature of the problem.

7. For $k>6$, $n=2$, and $N=k$. From $N=\frac{4k}{2n-nk+2k}$, it is easily seen that, for $n=2$, $N=k$. This includes, as well, the six cases of lunes already considered. From the denominator $2n-nk+2k$ it is seen that $2n+2k$ must be greater than nk: $2n+2k>nk$.

Therefore, $n(k-2)<2k$, whence $n<\frac{2k}{k-2}=2+\frac{4}{k-2}$. Now it is clear that for $k>6$, $n<3$. That is, n must be 2 or 1. But the values of n have both been considered.

Table of Results:

$k=1$, $n=2$, $N=1$ . . .	Lune.
$k=2$, n, any value, $N=2$	Hemispheres.
$k=3$, $n=2$, $N=3$ . . .	Lunes.
$k=3$, $n=3$, $N=4$ . . .	Triangles...*A*.
$k=3$, $n=4$, $N=6$ . . .	Quadrangles...*B*.
$k=3$, $n=5$, $N=12$ . . .	Pentagons...*C*.
$k=4$, $n=2$, $N=4$ . . .	Lunes.
$k=4$, $n=3$, $N=8$ . . .	Triangles...*D*.
$k=5$, $n=2$, $N=5$ . . .	Lunes.
$k=5$, $n=3$, $N=20$ . . .	Triangles...*E*.
$k\geqq 6$, $n=2$, $N=k$ . . .	Lunes.

From *A*, *B*, *C*, *D*, and *E*, we may easily pass to the proof that there are five and only five regular curvex polyhedrons.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

316. Proposed by B. F. FINKEL, Ph. D.

Prove that $\sum\limits_{r=1}^{r=n}(-1)^{n-1}\frac{1}{n}{}_nC_r=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ where ${}_nC_r=\frac{n(n-1)\ldots.(n-r+1)}{1.2.3\ldots.r}$. Dickson's *College Algebra*, ex. 13, p. 92.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The first member of the equation should be $\sum\limits_{r=1}^{r=n}(-1)^{r-1}\frac{1}{r}{}_nC_r$ instead of $\sum\limits_{r=1}^{r=n}(-1)^{n-1}\frac{1}{n}{}_nC_r$.

$$S_n=\sum_{r=1}^{r=n}(-1)^{r-1}\frac{1}{r}{}_nC_r=n-\frac{n(n-1)}{2.2!}+\frac{n(n-1)(n-2)}{3.3!}-\text{ to } n \text{ terms.}$$

$$S_{n+1}=(n+1)-\frac{(n+1)n}{2.2!}+\frac{(n+1)(n-1)}{3.3!}-\text{ to } n+1 \text{ terms.}$$

$$\therefore S_{n+1}-S_n=1-\frac{n}{2\,!}+\frac{n(n-1)}{3\,!}-\text{ to } n+1 \text{ terms}=\frac{1}{n+1}[1-(1-1)^{n+1}]=\frac{1}{n+1}.$$

$$S_1=1,\ S_2-S_1=\tfrac{1}{2} \text{ or } S_2=1+\tfrac{1}{2},$$
$$S_3-S_2=\tfrac{1}{3} \text{ or } S_3=1+\tfrac{1}{2}+\tfrac{1}{3},$$
$$.\quad .\quad .\quad .\quad .\quad .\quad .\quad .$$
$$S_n-S_{n-1}=1/n \text{ or } S_n=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+1/n.$$

(Vol. VII, No. 4, of the MONTHLY, page 105, line 8, gives $S_{n+1}-S_n=\frac{1}{1+n}$ by substituting -1 for x.)

Also solved by S. Lefsehetz.

317. Proposed by FRANCIS RUST, Allegheny, Pa.

Once, in classic days, Silenus lay asleep; a goat skin filled with wine near him. Dionysius passing by, profited, by siezing the skin, and drinking for two-thirds ($\frac{2}{3}$) of that time in which Silenus alone could have emptied said skin. At this point Silenus awoke, and seeing what was happening, snatched away the precious skin, and finished it.

Now, had both started together, and drank simultaneously, they would have consumed the wine skin in two hours less time. And, in this case, Dionysius' share would have been one-half as much as Silenus did secure, by waking and snatching the skin.

In what time would either one of them alone finish the goat-skin?

Solution by PROFESSOR F. L. GRIFFIN, Ph. D., Williams College.

Let x=fractional part which S drank, and y=number of hours S requires for entire skin. Then $\frac{2}{3}y$=time D was drinking, and xy=time S was drinking; $y(\frac{2}{3}+x)$ = time they used consecutively. Also, since $\frac{3(1-x)}{2y}$=part D drinks per hour, or $\frac{3(1-x)+2}{2y}$=part both drink per hour, the time required when drinking simultaneously $=\frac{2y}{5-3x}$.

Hence, (A) $y(\frac{2}{3}+x)=2+\frac{2y}{5-3x}$.

Again, the part D would get when they drink simultaneously $=\left(\frac{2y}{5-3x}\right)\frac{3(1-x)}{2y}$, or $\frac{3-3x}{5-3x}$; hence, by the problem, (B) $\frac{3-3x}{5-3x}=\frac{1}{2}x$.

Equation (B) gives $x=\frac{2}{3}$ or 3, the latter value being impossible.

Then (A) becomes $\frac{4}{3}y=2+\frac{2}{3}y$, or $y=3$; and since D drinks $\frac{1}{6}$ per hour, his time would be 6 hours.

Also solved by V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

318. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Sum to infinity the series $n/(4n^2-1)^2$ beginning with $n=1$.

Solution by J. W. CLAWSON, Ursinus College, Collegeville, Pa.; HOWARD C. FEEMSTER, A. B., York College, York, Neb.; J. EDWARD SANDERS, Weather Bureau, Chicago, Ill., and S. LEFSEHETZ, Pittsburg, Pa.

$$\sum_{n=1}^{n=\infty}\frac{n}{(4n^2-1)^2}=\sum_{n=1}^{n=\infty}\frac{1}{8}\left[\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right]$$

$$=\underset{n=\infty}{\text{Lt.}}\,\frac{1}{8}\left[\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\frac{1}{(2n-1)^2}\right)\right.$$

$$\left.-\left(\frac{1}{3^2}+\frac{1}{5^2}+\dots\frac{1}{(2n-1)^2}+\frac{1}{(2n-1)^2}\right)\right]=\underset{n=\infty}{\text{Lt.}}\,\frac{1}{8}\left[\frac{1}{1^2}-\frac{1}{(2n+1)^2}\right]=\frac{1}{8}.$$

Also solved by V. M. Spunar, G. B. M. Zerr, J. Scheffer, S. A. Corey, and T. J. Fitzpatrick.

GEOMETRY.

343. Proposed by O. J. BROWN, Fairhope, Ala.

From any external point of a triangle, to draw a line so as to divide the triangle into two equal parts.

I. Solution by C. N. SCHMALL, New York City.

Construction: Let P be the given point and ABC the given triangle. Join D, E, F, the middle points of the sides. Now draw PGI parallel to BC and join EG. From D draw DH parallel to GE, and then draw HI parallel to BA and meeting PG prolonged in I. On PI as a diameter describe a semi-circle and thereon lay off $PK=PG$. Draw KI, and in the base BC lay of $HM=KI$. Draw PM cutting AB in L. The line PLM bisects the triangle and is the line required.

Proof: Let PM and HI meet in N. Now the triangles PGL, PIN, MHN are clearly similar.

Also, since $PI^2=PK^2+KI^2=PG^2+HM^2$ (for $PK=PG$, and $KI=HM$).

$\therefore \triangle PIN=\triangle PGL+\triangle MHN \ldots (1)$.

Hence, the quadrilateral $LGIN$ is equal to the triangle MHN;

$\therefore$ triangle BML=parallelogram $BHIG\ldots (2)$.

Again, GE and DH are parallel;

$\therefore GB : BE = DB : BH \ldots (3)$.

Hence, the parallelograms $BHIG$ and $BEFD$ have a common angle B, and the including sides are reciprocally proportional.

$\therefore$ by (3), parallelogram $BHIG$=quadrilateral $BEFD$.

But by (2), parallelogram $BHIG$=triangle $BML\ldots (2)$.

$\therefore$ triangle BML=parallelogram $BEFD=\frac{1}{2}$ triangle ABC.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let P be the given point. Draw PE parallel to AB, cutting AC in D; make parallelogram $DEFA=\frac{1}{2}$ triangle ABC. On AB at F erect perpendicular $FG=PD$, and make $GA=PE$. Connect P with Q, then PQ will be the required line which bisects triangle ABC.

For, $\triangle FHQ : \triangle PHE=FQ^2 : PE^2=PE^2-PD^2 : PE^2$.

$$\therefore \frac{\triangle FHQ}{\triangle PHE}=1-\frac{PD^2}{PE^2}=1-\frac{\triangle PDH}{\triangle PEH}.$$

$\therefore \triangle FHQ=\triangle PEH-\triangle PDI=DIHE$; $\therefore FHQ+IHFQ=DIHE+IHFA$ $=DEFA$.

Note: To construct $DEFA$, connect C with the mid-point M of AB, draw DM, and CN parallel to DM, then F will be the mid-point of AN.

Also solved by G. B. M. Zerr, V. M. Spunar, and Daniel B. Northrup.

NOTE. The following gentlemen should have received credit for solving 342: J. A. Caparo, V. M. Spunar, and S. Lefsehetz.

CALCULUS.

274. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

About the vertices of a regular tetrahedron four spheres are drawn with radii equal to the edge of the tetrahedron. Find the volume common to them all.

Solution without the use of the Calculus by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

One vertex of the tetrahedron is at the center of each sphere while the other three vertices are in the surface.

Hence the total volume common to all the spheres = the volume of the tetrahedron + the volumes of four equal spherical segments.

Let a = the edge of the tetrahedron and also the radius of each sphere. Then $\frac{1}{3}a\sqrt{6}$ = altitude of tetrahedron, and $\frac{1}{12}a^3\sqrt{2}$ = its volume.

$a-\frac{1}{3}a\sqrt{6}=\dfrac{a}{3}(3-\sqrt{6}$ = altitude of segment. $\dfrac{\pi a^3}{81}(3-\sqrt{6})^2(6+\sqrt{6})=$ $\dfrac{\pi a^3}{27}(18-7\sqrt{6})$ = volume of spherical segment.

Total volume required $=V=\dfrac{4\pi a^3}{27}(18-7\sqrt{6})+\dfrac{a^3\sqrt{2}}{12}$.

$\therefore V\dfrac{a^3}{108}(9\sqrt{2}+288\pi-112\pi\sqrt{6})$.

This solid might be called a spherical tetrahedron.

275. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Explain fully why the circular measure of an angle is used in the calculus.

No satisfactory answer of this problem has been received.

276. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In a certain country the tax per \$1 on a person's income varies as the cube root of the number of dollars, and when the income is \$8000 the rate per dollar is 5 cents. Find the largest net income possible.

Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let x = the number of dollars, gross income; y = the rate of taxation on \$1; xy = the tax on x dollars; and u = the net income; from which we have $u=x-xy$... (1).

From the problem we have, $(8000)^{\frac{1}{3}}:x^{\frac{1}{3}}::0.05:y$; or $y=x^{\frac{1}{3}}/400$... (2); and from (2), $xy=x^{\frac{4}{3}}/400$... (3). Substitute in (1) the value of xy from (3), and we have, $u=x-x^{\frac{4}{3}}/400$... (4). Differentiate (4) and we have, $du/dx=1-x^{\frac{1}{3}}/300=0$; or x = \$27,000,000. Substitute this value of x in (2), and we have, $y=\frac{3}{4}$, the rate of taxation. Therefore the net income is $\frac{1}{4}$; or $\frac{1}{4}$ of \$27,000,000 = \$6,750,000, the maximum net income.

To find when the rate of taxation is prohibitive, let z = the gross income, and we have, $(8000)^{\frac{1}{3}}:z^{\frac{1}{3}}::5\%:100\%$; or z = \$64,000,000. That is to say, when the gross income is \$64,000,000, the whole income is taken for the tax.

Also solved by H. C. Feemster, J. Scheffer, J. E. Sanders, V. M. Spunar, S. A. Corey, J. W. Clawson, and the Proposer.

MECHANICS.

226. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A frustum of a cone, vertical angle α, is cut off by two spheres whose centers are the vertex. The radius of one sphere is n times that of the other, and the density of the cone varies as the distance of the vertex. Find the ratio into which the centroid of the frustum divides the axis.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let R=radius of smaller sphere, nR=radius of larger sphere. Take the axis of the cone as that of z. Then $\bar{x}=\bar{y}=0$.

$$\bar{z}=\frac{\int\int\int r^4 \sin\theta\cos\theta\, d\theta\, d\phi\, dr}{\int\int\int r^3 \sin\theta\, d\theta\, d\phi\, dr}.$$

By frustum of cone here is evidently meant that part of the cone between the spheres.

$$\therefore \bar{z}=\frac{\int_0^{\frac{1}{2}\alpha}\int_0^{2\pi}\int_R^{nR} r^4\sin\theta\cos\theta\, d\theta\, d\phi\, dr}{\int_0^{\frac{1}{2}\alpha}\int_0^{2\pi}\int_R^{Rn} r^3\sin\theta\, d\theta\, d\phi\, dr}=\tfrac{4}{5}R\left(\frac{n^5-1}{n^4-1}\right)\frac{\int_0^{\frac{1}{2}\alpha}\sin\theta\cos\theta\, d\theta}{\int_0^{\frac{1}{2}\alpha}\sin\theta\, d\theta}$$

$$=\tfrac{2}{5}R\left(\frac{n^5-1}{n^4-1}\right)(1+\cos\tfrac{1}{2}\alpha).$$

$$\frac{nR-\bar{z}}{\bar{z}-R}=\frac{5n(n^4-1)-2(n^5-1)(1+\cos\frac{1}{2}\alpha)}{2(n^5-1)(1+\cos\frac{1}{2}\alpha)-5(n^4-1)}.$$

If planes were passed through the intersections of the spheres with the cone, then $R\cos\frac{1}{2}\alpha\,(n-1)$ is the height of the frustum, and the limits of r are $nR\cos\frac{1}{2}\alpha\sec\theta=R_1$, and $R\cos\frac{1}{2}\alpha\sec\theta=R_2$.

$$\therefore \bar{z}=\tfrac{4}{5}R\cos\tfrac{1}{2}\alpha\left(\frac{n^5-1}{n^4-1}\right)\frac{\int_0^{\frac{1}{2}\alpha}\sin\theta\cos\theta\sec^5\theta\, d\theta}{\int_0^{\frac{1}{2}\alpha}\sin\theta\sec^4\theta\, d\theta}=\tfrac{4}{5}\left(\frac{n^5-1}{n^4-1}\right)R\cos\tfrac{1}{2}\alpha.$$

$$\therefore \frac{nR\cos\frac{1}{2}\alpha-\bar{z}}{\bar{z}-R\cos\frac{1}{2}\alpha}=\frac{n^5-5n+4}{4n^5-5n^4+1}.$$

II. **Solution by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.**

Let α=one-half the vertical angle of cone, a, and an radii of spheres. With the vertex of the cone as origin, and the axis of y as the axis of the cone, the mass of the frustum contained between the spheres is

$$M=k\pi\tan^2\alpha\int_{a\cos\alpha}^{na\cos\alpha}y^3dy+k\pi\int_{an\cos\alpha}^{an}(a^2-y^2)ydy-k\pi\int_{a\cos\alpha}^{a}(a^2-y^2)ydy$$

$$=\frac{\pi ka^4(n^4-1)\sin^2\alpha}{4}.$$

These integrals represent the mass of a frustum whose bases are planes passing through the line of intersection of the surfaces, and the masses of the spherical segments added and subtracted from the above mass. Call these masses M_c, M_a, M_n, and $\bar{y}_c$, $\bar{y}_a$, $\bar{y}_n$, distances of their centroids from origin.

Then, $M_c\bar{y}_c=\pi k\tan^2\alpha\int_{a\cos\alpha}^{na\cos\alpha}y^4dy=\frac{1}{5}\pi ka^5\cos^5\alpha\tan^2\alpha\,(n^5-1).$

$$M_a\bar{y}_a=\pi k\int_{a\cos\alpha}^{a}(a^2-y^2)y^2dy=\frac{\pi ka^5}{15}(2-5\cos^3\alpha+3\cos^5\alpha).$$

$$M_n\bar{y}_n=\pi k\int_{an\cos\alpha}^{an}(a^2-y^2)y^2dy=\frac{\pi ka^5n^3}{15}[5(1-\cos^3\alpha)-3n^2(1-\cos^5\alpha)].$$

But $M\bar{y}=M_c\bar{y}_c+M_n\bar{y}_n-M_a\bar{y}_a$.

$$\therefore\bar{y}=\frac{4a[2+5n^3-3n^5+n^3\cos^3\alpha(3n^2+5)+2\cos^3(4+3\cos^2\alpha)]}{15(n^4-1)\sin^2\alpha}.$$

227. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

Solution by the PROPOSER.

Let $AB=2a$ be the tunnel; $OA=OB=R$; C the mid-point of AB; P and R, points between A and C; $OP=y$, $PC=x$, $CR=d$, $\angle POC=\theta$; f the acceleration at P; f' the acceleration along PC at P, and R the starting point of the sphere. Beneath the earth's surface, the acceleration varies directly as the distance from the center.

$\therefore f:g=y:R$ or $f=gy/R$. $f':f=\sin\theta:1$ or $f'=f\sin\theta=gy\sin\theta/R$.
But $y=x/\sin\theta$. $\therefore f'=gx/R$.

I. If the tunnel is perfectly smooth, the equation of motion is $d^2/dt^2 + gx/R = 0$. $\therefore (dx/dt)^2 + gx^2/R = C$. Since $t=0$ when $x=d$, $C = gd^2/R$.

$\therefore (dx/dt)^2 = (g/R)(d^2 - x^2) = (\text{velocity})^2 = v^2$. When the sphere arrives at C, $x=0$. $\therefore v = d\sqrt{(g/R)} = a\sqrt{(g/R)}$. When the sphere starts at A,

$$t = \sqrt{\frac{R}{g}}\int_0^m \frac{dx}{\sqrt{(d^2-x^2)}} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{m}{d}. \quad \text{If } m=d,\ t=\frac{\pi}{2}\sqrt{\frac{R}{g}}.$$

As this expression is independent of the distance from C, the time for the sphere to move from A to C or from any point between A and C to C is the same. The time is the same for any tunnel, and is the same as the time required for the sphere to fall from the surface to the center, or from any point beneath the surface to the center of the earth.

Let $R=20902410$ feet, $g=32.10614$ feet.

$\therefore \sqrt{(R/g)} = 806.871$, and therefore $t = (\frac{1}{2}\pi)\sqrt{(R/g)} = 1267.433$ seconds $=21$ minutes, 7.433 seconds.

II. When the tunnel is perfectly rough. Let F be the friction. Then the equations of motion are $d^2x/dt^2 + gx/R + F = 0$, $k^2d^2\phi/dt^2 = bF$, $b\phi = x$ or $bd\,\phi = dx$, $bd^2\phi = d^2x$.

$$\therefore \left(\frac{k^2}{b^2}\right)\frac{d^2x}{dt^2} = F;\ \therefore \left(\frac{b^2+k^2}{b^2}\right)\frac{d^2x}{dt^2} + \frac{gx}{R} = 0;\ \frac{d^2x}{dt^2} + \frac{5gx}{7R} = 0;\ \left(\frac{dx}{dt}\right)^2 = v^2 = \frac{5g}{7R}(d^2-x^2).$$

$\therefore v = d\sqrt{(5g/7R)}$, at the point C; $v = a\sqrt{(5g/7R)}$, when the sphere starts at A.

$$t = \sqrt{\frac{7R}{5g}}\int_0^d \frac{dx}{\sqrt{(d^2-x^2)}} = \frac{\pi}{2}\sqrt{\frac{7R}{5g}} = 24 \text{ minutes, } 59.627 \text{ seconds.}$$

Therefore the time is the same for any perfectly rough tunnel.

If the sphere starts with a velocity v_1^2, then for a perfectly smooth tunnel,

$$v^2 = \left(\frac{dx}{dt}\right)^2 = \frac{g}{R}(a^2-x^2) + v_1^2,$$

supposing the sphere to start from A with the velocity v_1.

$$\therefore t = \sqrt{R}\int_0^a \frac{dx}{\sqrt{[Rv_1^2 + g(a^2-x^2)]}} = \sqrt{\frac{R}{g}}\sin^{-1}\frac{a\sqrt{g}}{\sqrt{(Rv_1^2+ga^2)}}.$$

This equals the time to go from A to C.

For a perfectly rough tunnel, $v^2=(dx/dt)^2=(5g/7R)(a^2-x^2)+v_1^2$.

$$\therefore t=\sqrt{7R}\int_0^a \frac{dx}{\sqrt{[7Rv_1^2+5g(a^2-x^2)]}}=\sqrt{\frac{7R}{5g}}\sin^{-1}\frac{a\sqrt{5g}}{\sqrt{(7Rv_1^2+5ga^2)}}.$$

III. If the tunnel extends beyond the surface of the earth: then let SB be the tunnel. Above the earth's surface, acceleration varies inversely as the square of the distance from the center.

Let $CO=c$, $OQ=y$, $CQ=x$, $SC=h$. Then $f:g=R^2:y^2$ or $f=gR^2/y^2=gR^2/(c^2+x^2)$. $f':f=\sin\theta:1$. $\therefore f'=f\sin\theta=fx/\sqrt{(c^2+x^2)}$.

$$\therefore f'=\frac{gR^2x}{\sqrt{[(c^2+x^2)^3]}}.\quad \therefore \frac{d^2x}{dt^2}+\frac{gR^2x}{\sqrt{[(c^2+x^2)^3]}}=0.$$

$$\therefore \left(\frac{dx}{dt}\right)^2=2gR^2\left[\frac{1}{\sqrt{(c^2+x^2)}}-\frac{1}{\sqrt{(c^2+h^2)}}\right]=v_1^2.$$

$$\therefore v_1=R\sqrt{2g\left[\frac{1}{\sqrt{(a^2+c^2)}}-\frac{1}{\sqrt{(h^2+c^2)}}\right]}=R\sqrt{\frac{2g}{R}-\frac{2g}{\sqrt{(c^2+h^2)}}}.$$

When the sphere reaches A,

$$t=\frac{\sqrt[4]{(c^2+h^2)}}{R\sqrt{(2g)}}\int_0^h \frac{\sqrt[4]{(c^2+x^2)}\,dx}{\sqrt{[\sqrt{(c^2+h^2)}-\sqrt{(c^2+x^2)}]}}.$$

$$\therefore t=\frac{2(c^2+h^2)^{\frac{5}{4}}}{R\sqrt{(2g)}}\int_{\phi_1}^{\frac{1}{2}\pi}\frac{\sin^4\phi\,d\phi}{\sqrt{[(c^2+h^2)\sin^4\phi-c^2]}},$$

where $c^2+x^2=(c^2+h^2)\sin^4\phi$ and $\phi_1=\sin^{-1}\sqrt[4]{\frac{R^2}{c^2+h^2}}$.

t=time to move from S to A. For a perfectly rough tunnel, $v_2=\sqrt{(\frac{5}{7})}v_1$; $t_1=\sqrt{(\frac{7}{5})}t$.

Also solved by S. Lefseletz.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

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160. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that the integer next above $(1+\sqrt{3})^{2n}$ is divisible by 2^{n+1}.

Proof by G. E. WAHLIN, Urbana, Ill.

$|1-\sqrt{3}|<1$. Hence (1) $0<(1-\sqrt{3})^{2n}<1$.

But $(1+\sqrt{3})^{2n}+(1-\sqrt{3})^{2n}=E$, a rational integer, and since (1) is true, this is the integer next above $(1+\sqrt{3})^{2n}$.

Since $\dfrac{(1\pm\sqrt{3})^2}{2}=2\pm\sqrt{3}$, we have $\dfrac{(1\pm\sqrt{3})^{2n}}{2^n}=(2\pm\sqrt{3})^n=P\pm Q\sqrt{3}$, P and Q rational integers.

Then $\dfrac{(1+\sqrt{3})^{2n}}{2^n}+\dfrac{(1-\sqrt{3})^{2n}}{2^n}=\dfrac{E}{2^n}=2P$. Therefore, $\dfrac{E}{2^{n+1}}=P$, and hence E is divisible by 2^{n+1}.

Also solved by J. Scheffer, G. B. M. Zerr, and V. M. Spunar.

PROBLEMS FOR SOLUTION.

ALGEBRA.

321. Proposed by C. C. BLAND, Attorney at Law, Rolla, Mo.

A corporation is capitalized for \$20,000. 125 shares of the par value of \$100 per share has been issued. A has 27 19/78 shares. B, C, D, E and F each have 19 43/78 shares. It is the wish of the corporation to cancel the certificates held by A, B, C, D, E, and F, and to issue new certificates to each of them in lieu of those now held by them, and to avoid the issuance of any certificate for a fraction of a share. How many shares should each receive, the whole not to exceed 200, at the same time maintaining the present interest of each in the corporation?

322. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Take six consecutive prime numbers, as 53, 59, 61, 67, 71, and 73, and find the least whole number such that if it be divided by 59 the remainder will be 53, if it be divided by 67 the remainder will be 61, and if it be divided by 73 the remainder will be 71, and show that this least whole number and the succeeding consecutive whole numbers that will fulfill this condition as to divisions and remainders are in arithmetical progression; and also show whether or not this is a general law for n consecutive prime numbers; and if there be such a general law whether or not that general law will lead to a general law for the finding of prime numbers.

323. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that the relation $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$ can hold for real numbers only when they are in proportion.

GEOMETRY.

348. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two parabolas and a rectangular hyperbola circumscribe a given quadrilateral. Find a relation between the squares of the latera recta of the parabolas and the squares of the perpendiculars from the center of the hyperbola to the axes of the parabolas.

349. Proposed by J. A. CAPARO, Notre Dame University, Notre Dame, Indiana.

Given the radius of a circular smooth cylinder and its position with respect to a source of light and the eye. Find a geometrical construction to determine the line of brilliancy.

350. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Given the quadrilateral $AB=a=225$, $BC=b=153$, $CD=c=207$, $DA=d=135$, $AC=e=240$. Find the side of the square inscribed in this quadrilateral having a corner in each side.

CALCULUS.

279. Proposed by L. H. McDONALD, M. A., Ph. D., Sometime Tutor at Cambridge, Jersey City, N. J.

Find the ellipse of minimum area which will pass through the vertices of a triangle. (Hedrick-Goursat's *Math. Anal.*, p. 133, ex. 9.)

280. Proposed by C. N. SCHMALL, 89 Columbia Street, New York.

Find the envelope of the system of spheres

$$\left.\begin{array}{r}(x-a)^2+(y-b)^2+z^2=r^2\\ a^2+b^2=c^2\end{array}\right\}.$$

281. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that if n be a positive integer greater than unity,

$$\text{Log}n=(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{2n})-C+\frac{B_1}{2n^2}-\frac{B_2}{4n^4}+\frac{B_3}{6n^6}-\frac{B_4}{8n^8}+\ldots (1).$$

REMARK.—This development may be obtained by employing the formula given by the proposer in *Annals of Mathematics*, second series, Vol. 5, No. 4, July, 1904, but other proofs are also desired. It will be observed that (1) offers a ready method of evaluating C, which is remarkably simple and very rapidly convergent if $n>$ or $=10$. Compare with method given by Mr. Bromwitch in *Messenger of Mathematics*, October, 1906.

MECHANICS.

233. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A shot, mass m, is fired from a gun, mass M, which can move on a horizontal plane. The muzzle velocity of the shot is given, the muzzle always points in the same direction as the shot leaves it. The envelope of the trajectories is a parabola. Find the ratio of the distances of its focus and directrix from the plane in terms of M and m.

234. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

A ladder is placed with one end resting against a smooth wall and making with it an angle ϕ. Also, the roughness of the ground prevents it from slipping. A man weighing as much as the ladder ascends to the top. Taking μ as the coefficient of friction, prove:

(a) The ladder will slip before he gets to the top if $\phi>\tan^{-1}4\mu/3$.

(b) If the ascent be feasible, there will be three times as much friction when he is at the top as when he is at the bottom, (See Jeans' *Theoretical Mechanics*, p. 47.)

235. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A uniform heavy rod turns freely round a hinge at one end and rests with the other against a rough vertical wall, at angle, α, to the wall. Find the angle of arc on which this end may rest, and the pressures at the ends of the arc.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

165. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Factor (if possible), 11, 111, 111, 111.

166. Proposed by H. S. VANDIVER, Bala, Pa.

Eliminate any five of the seven quantities h, j, k, l, m, r, s, from:

$$j+m+r=k+l+s,$$
$$h=3(r+s)-(n+1),$$
$$j+m+2r+s=n,$$
$$hr+j^2+km+ms=jk+mr+s^2+rs,$$
$$ks+jl+rs+r^2=sh+jk+l^2+rk,$$
$$hr+jl+ks+ls+km=jm+mr+2rs+rj.$$

167. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Prove that $\Pi\dfrac{p+(-1)^{\frac{1}{2}(p+1)}}{p-(-1)^{\frac{1}{2}(p+1)}}=2$, the consecutive values of p being the natural odd primes in order.

AVERAGE AND PROBABILITY.

204. Proposed by F. P. MATZ, Reading, Pa.

On a random chord in a circle two points are taken at random. What is the chance a second chord drawn at random will pass between the two points?

205. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

What is the probability that the triangle formed by joining three points, one at randam in each of three equal circles, mutually tangent, has an obtuse angle?

NOTES AND NEWS.

Professor L. E. Dickson left on September 4th for Europe, where he will spend the year in study and travel. F.

Professor R. D. Carmichael has been granted a year's leave of absence and is now studying mathematics at Princeton. F.

Professor B. F. Yanney, of Mount Union College, Alliance, Ohio, has been granted a year's leave of absence. He will spend the year in study at the University of Chicago. F.

Professor W. A. Manning of Stanford University, secretary of the San Francisco Section of the American Mathematical Society, has joined the mathematical faculty of the University of Illinois for one year in exchange with E. W. Ponzer. Professor Manning is the author of various articles on substitution groups. M.

On July 11th occurred the death of Professor Simon Newcomb, probably the most distinguished astronomer that America ever produced. Professor Newcomb was also recognized as a mathematician of the first rank and was honored the world over for his great scientific achievements. For a biography of Professor Newcomb, see THE AMERICAN MATHEMATICAL MONTHLY, Vol. I, pp. 253-256. F.

The seventy-ninth annual meeting of the British Association for the Advancement of Science was held at Winnipeg, Canada, under the presidency of J. J. Thomson from August 25 to September 1, 1909. Fourteen hundred members and associates were in attendance, about five hundred coming from Europe and about one hundred and fifty from the United States. The papers devoted to pure mathematics were as follows:

E. H. Moore, Theorems in General Analysis; E. W. Hobson, F. R. S., On the Present Position of the Theory of Aggregates; G. A. Miller, Generalizations of the Icosahedral Group; G. A. Bliss, A New Proof of Weierstrass' Theorem; J. H. Grace, F. R. S., On Ideal Numbers; P. A. MacMahon, F. R. S., On a Correspondence in the Theory of the Partition of Numbers; W. H. Metzler, On a Continuant Expressed as the Product of Linear Factors; Ellery W. Davis, Imaginary Geometry of the Conic; Florian Cajori, On the Invention of the Slide Rule; J. W. Nicholson, The Asymptotic Expansion of Legendre Functions. M.

BOOKS.

Coordinate Geometry. By Henry Burchard Fine and Henry Dallas Thompson. 8vo. Red Cloth Sides and Leather Back. viii+300 pages. New York: The Macmillan Co.

This book, which has the advantage of having been tested out in class room use for three years, will appeal strongly to all good teachers of Analytical Geometry because of the scholarly presentation of the subject. While it does not begin with the easiest possible concepts leading up to the subject, yet it is believed to be sufficiently elementary in its initial steps to enable the earnest student to easily master it. The treatment of higher plane curves is very brief, and it is to be regretted that brief historical notes concerning those that are treated have been omitted.

Excellent engravings of some of the second degree surfaces are given in the last pages of the book. F.

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By Harold Crabtree, M. A., Formerly Scholar of Pembroke College, Cambridge. Assistant Master at Charterhouse. With Illustrations. 8vo. Cloth, xii+140 pages. Price, $1.50. New York: Longmans, Green & Co.

The author's object in writing this interesting book is to bring within the range of the abler mathematicians of the public schools of England and of the First Year Undergraduates of the universities, a subject which has been previously considered too difficult for any but the more advanced students in mathematics.

The story was told that when Felix Klein gave his lecture on the Top at Princeton, a

lady told Professor Klein that her little boy (about ten years old) was greatly interested in tops, and she thought he would enjoy a lecture on the subject from so distinguished a scholar. The reporter failed to tell how the youngster entertained himself during the lecture, but if he did not have to be aroused from his slumbers by some vigorous shaking at its close he surely must have been a very unusual lad.

The book before us is not intended, it must be understood, for popular reading, nor for the amusement of boys interested in tops. It does, however, put the theory of tops in a mathematical dress easily recognized by any one having a good working knowledge of the Calculus, and a clear understanding of a few fundamental laws of Mechanics. The book is well written, and will be of interest to the inventive mind as well as to the mathematician. F.

A Treatise on Differential Geometry. By Luther Pfahler Eisenhart, Preceptor in Mathematics, Princeton University. 8vo. Cloth. 474 pages. With Diagrams. Price, $4.50. Boston and Chicago: Ginn & Co.

It is the purpose of this book to introduce the student to the methods of differential geometry and to the theory of curves and surfaces developed thereby, to such an extent that he will be prepared to read the most extensive foreign treatises and journal articles. The reader is suppose to possess a knowledge of the calculus, elementary differential equations, and the elements of coordinate geometry of three dimensions. Hence the first half of the book may be used with seniors, and the remainder will constitute a full-year course for graduate students.

The method generally used is that of Gauss, common among German and Italian writers, but the kinematical method, frequently adopted in France, has been developed and applied where more feasible. This has been done not only because it furnishes the student with a powerful operator, but for the reason, also, that it develops geometrical thinking.

There are several hundred problems, some of which are direct applications of the accompanying sections, but many are theorems which might properly be established in a more extensive treatise. These have been inserted as an incentive to research and as preparation for larger problems.

The Integrals of Mechanics. By Oliver Clarence Lester, Professor of Physics in the University of Colorado; formerly Instructor in Physics in the Sheffield Scientific School, Yale University. 8vo. Cloth, 67 pages. With Diagrams. Price, 80 cents. Boston and Chicago: Ginn & Co.

The aim of this book is to furnish the conclusion to courses in the Integral Calculus such as are usually given in colleges and technical schools, and at the same time to provide for the beginning of Theoretical Mechanics, which usually follows the Calculus. The subject-matter is concerned entirely with such applications of the Calculus as the calculation of lengths, areas, volumes, densities, centers of mass, moments of inertia, and ellipsoids of inertia. These subjects are treated in great detail, all principles being fully illustrated by examples worked out in the text and by numerous problems set as exercises.

Since the ground covered is common to both the Integral Calculus and to Theoretical Mechanics, the author hopes in this way to save both time and energy; to save time by providing applications of the Calculus useful in Mechanics; to save energy by treating the purely mathematical parts of Mechanics entirely apart form the ideas of force and motion. This method avoids breaks in the continuity of the Mechanics course proper, and minimizes the liability of the student to such troublesome confusions as moment of inertia with the moment of a force, or center of gravity with the force of gravity.

Vector Analysis. An Introduction to Vector-Methods and Their Various Applications to Physics and Mathematics. By Joseph George Coffin, B. S., Ph. D., Instructor in Physics at the College of the City of New York. 12mo, Cloth. xix+248 pages, 69 figures. Price, $2.50 net. New York: John Wiley & Sons.

The first part of the book is devoted to a concise treatment of the fundamental principles of the subject, the remaining chapters, to the application of the analysis to the *beginnings* of mathematical physics, including geometry, mechanics, magnetism, electricity, heat and hydrodynamics. It was found necessary to omit many beautiful applications in elasticity, electron theory and other parts of physics in order to keep the size of the volume within bounds.

The student who takes up the later chapters, is supposed to be familiar, to a certain extent, with the subjects therein contained, and these chapters are intended to show the beginner how to translate and demonstrate the theorems into the new calculus. The writer therefore makes this his apology for a certain necessary lack of logical sequence in the treatment of the various subjects.

The treatment of alternating currents and allied subjects has been omitted, because in practically every modern book on the subject of notation of the special vector method employed, is fully explained in some part of it.

The notation adopted is that of Prof. Willard Gibbs, one of the too few great American physicists and mathematicians. The reasons leading to this choice are fully set forth in the Appendix. From *Preface.*

Complete Arithmetic. By George Wentworth and David Eugene Smith. 12mo. Cloth. Illustrated. vi+474 pages. Price, 60 cents. New York and Chicago: Ginn & Co.

The *Complete Arithmetic* is a text-book for grammar-school grades, thoroughly modern in spirit and material and arranged according to a topical plan.

The keynote of the method—present the reason briefly but clearly, then furnish such an amount of practice that the pupil cannot forget the principle—is that which has made the Wentworth texts the standard for a generation. Theory is reduced to a minimum and practice is abundantly provided in more than six thousand carefully graded problems and examples, all absolutely new.

The work conforms with modern business customes, and the needs of the future citizen are constantly kept in mind. The topics properly close with sets of exercises that relate to the vocational interests of our country, to the end that pupils may leave the *study* of arithmetic with the real *applications* of arithmetic clearly in mind.

The book contains all the topics ordinarily studied after a primary arithmetic has been completed, and omits such subjects as are too technical or have become obsolete.

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THE FOURTH DIMENSION.

By E. J. WILCZYNSKI, University of Illinois.

In order to describe the position of a point upon a line, we usually proceed as follows. We place a scale, like that of a thermometer, next to the line and make note of the scale-reading which corresponds to the point considered. As in the case of the thermometer this scale reading will be positive or negative according as the point is situated on one side or the other of the zero point of the scale. It thus becomes possible to characterize the position of a point of a line by means of a *single* positive or negative number. This is what is meant by the statement that a line has only *one dimension.*

If we wish to describe the position of a point on a surface, we shall have to employ two numbers. A familiar illustration of this fact is given by the ordinary method of describing the position of a point on the earth's surface by means of its latitude and longitude. These two numbers are called the coordinates of the point. The fact that *two* numbers are required is equivalent to the statement that the surface has *two dimensions.*

But if we take into account the fact that there are mountains and valleys, it becomes clear that a third coordinate becomes necessary which shall express the altitude of a point above, or its depression below sea-level. Obviously three numbers will suffice to determine the position of any point in space. This fact, expressed in another way, leads to the statement that *space has three dimensions.*

Nevertheless, this property of space of having three dimensions is not really a property of space at all, but depends upon our way of looking at it which, as we shall see, is largely due to physiological causes and has no logical foundation whatever. In fact, a number applied to an object of experience always implies that something has been taken as a unit. If we say that the length of a line is expressed by the number five, we can attach no meaning to such an assertion unless we know what unit of measurement has been employed. In the same way, the statement that space has three dimensions is incomplete and meaningless, although our above discussion may

be taken to supplement and explain its meaning. The reader may have noticed that we discussed only the method of determining *points* in space, as though space could contain nothing else. To be sure we *may* think of space as an aggregate of points, and analyze all geometric configurations into point-elements; but there is no *logical necessity* for doing so. Let us, for instance, think of space as the assemblage of all of its spheres. This does not change the nature of space in the least; the question is merely whether we prefer to analyse the configurations of geometry into point elements or into sphere elements. As a point-aggregate, space has three dimensions, because it requires three numbers (coordinates) to determine one of its elements (points); As a sphere-aggregate, space has four dimensions, because it requires four numbers (coordinates) to determine one of its elements (spheres). That this latter statement is true can be seen at once, since it takes three numbers (coordinates) to determine the center of a sphere, and a fourth to determine its radius.

A second illustration is of even greater importance. *Space, considered as an aggregate of straight lines, has four dimensions.* To prove this, it suffices to show that four numbers are required to determine a straight line in space. Let us choose any two planes in space, infinite in extent. Two numbers are required to determine a point in one of them, two more to determine a point in the other. The line determined by these two points, therefore, requires four numbers for its identification; *i. e.*, with the straight line as space-element, space has four dimensions.

It may be urged that it is absurd, useless, and artificial to regard space as anything except as an aggregate of points. That it is neither absurd nor useless is conclusively demonstrated by the fact that some very important branches of geometry, finding applications in such practical mattess as steel frame and bridge construction, owe a great part of their recent progress to this point of view. That it seems artificial to analyze the configurations of space into line-elements rather than point-elements, may be regarded as a mere accident of human physiology rather than as a matter of logic. That this is true will perhaps become apparent from the following argument.

Let us imagine a rational being whose body is incapable of motion. His space ideas will therefore be based entirely upon sight impressions, except in so far as he becomes conscious of the muscular efforts which accompany the act of adjusting his eyes to the proper focus for objects at different distances. These muscular sensations, combined with those which arise from the change in the angle of convergence of the two eyes when fixed upon objects at different distances, and the stereoscopic effect, will enable him to arrive at the idea of distance; that of direction is furnished directly by the sense of sight. Let us suppose, however, that his eyes, instead of being, like our own, approximately spherical, are cylindrical in shape. The image of a point upon his retina would be a line. Would it not be natural for him to take the line as his space-element rather than the point? If we

now endow him with sense organs in all other respects similar to our own, and also permit him to move about, the notion of a point would also present itself to him, in a secondary way however, and brought to his attention principally by the sense of touch. He would observe that the line image of a point rotates in a certain way when he inclines his head to the right or left; so that, by associating this fact with the idea of a point obtained by the sense of touch, he would come to look upon a point as that geometric configuration which is common to all of the lines which pass through it. He would analyze the point into line-elements, just as we analyze the line into point-elements.

We have indicated the existence of two distinct four-dimensional geometries without leaving the realm of ordinary space. But the question may be asked: is a space conceivable which, considered as an aggregate of *points*, has four dimensions? Probably it is this phase of the question of which most persons think when the subject of the fourth dimension comes under discussion.

That such a space is conceivable in a *logical, purely mathematical sense* is evident from what has been said. Abstractly, *i. e.*, without reference to concrete sense impressions, a point may be defined as any concept which is determined by a set of numbers. We, therefore, obtain at once a four-dimensional point-space in the abstract sense, by choosing the space-element of ordinary space in such a way as to give rise to a four-dimensional geometry, and then substituting the word *point* for the word *space-element.* If it be desired, howover, to obtain a more concrete illustration to show the possibility of the *objective existence* of a four dimensional point-aggregate, we shall be compelled to analyze somewhat more closely the psychological foundations of our space concept. This will lead to some speculations which, while highly imaginative, are not illogical.

It is a familiar notion that those properties of things which we call tone and color do not exist objectively. The vibrations of the air or of the ether, converted into nervous energy and transmitted to the brain, produce upon the mind the peculiar impressions of tone and color. Many philosophers claim that space, likewise, has no objective existence. The spacial relation, according to them, is to be looked upon as something which the mind creates in reacting upon the stimuli of the external world; the mind creates space as it does tone and color. If this is true, space is for us such as we see it; not because it *is* so, but because we *see* it so. If there existed a race of men, or even a single rational being whose space concept involved more than three dimensions, that new kind of space would have the same claim to existence as the old.

Since our argument clearly becomes meaningless on this basis, let us take the point of view that an absolute space exists, and that it has intrinsic properties which are not created by the mind. We shall attempt to show that, even on that assumption, beings somewhat differently organized from

ourselves could develop a space concept vastly different from our own. One such case has already been indicated.

The mere fact that we find it difficult to conceive of what the fourth dimension would "look like," no more disproves its existence than the inability of a blind man to conceive of color can be held to show that no such thing exists. It is not hard to imagine a race of beings (flat landers), living in a plane, to whom a third dimension might appear as mysterious as a fourth does to us. Suppose that such a flatlander were to surround an identifiable object of his infinitely thin world by a circle. Nothing can be more obvious to him than the proposition that this object can never reach the outside of the circle without crossing the circumference. But suppose that he sees it disappear, and reappear a moment later outside. To us there is nothing mysterious about this; the object has described a path in space. The flatlander, however, would have to disbelieve his senses, call to his assistance the supernatural, or adopt the hypothesis of a three-dimensional space, of which his own two-dimensional world is but a part. It has been proposed to explain some of the phenomena alleged to have been observed by the spiritualists on the assumption that space has four dimensions. A four dimensional being, for instance, could easily escape from a hermetically closed room. For him, a sphere is no more effective as a barrier than is a circle for us. We see then the possibility of obtaining experimental evidence for the objective existence of a fourth dimension, without wishing to assert that such evidence has already been furnished.

But we may look at the question in still another way. It can hardly be denied that our space concept would be much modified if our sense organs were different from what they are. What, in fact, are the experiences upon which our own ideas of space are based? Obviously the sense of sight is an important factor. It is clear, however, that our geometry would be two dimensional if we were stationary beings, rooted to the ground like a tree, provided with but a single immobile eye in focus for all distances, or if the act of accommodation were not accompanied by muscular sensations. We should be deprived of the notion of distance. The existence of an arm, an organ of touch, etc., enables us to enlarge our idea of space so as to include the notion of distance. Our space concept, then, is an abstraction which unites for us in a consistent manner, a number of qualitatively different sense impressions.

But we do not unite all of our sense-impressions in this way. The notion of color, for instance, does not differ qualitatively from that of direction even as much as does the notion of distance, being a result of pure sight impressions, while that of distance is not. It is quite conceivable that color is no more an intrinsic property of an object than is its distance from us. It is not absurd to imaginine that bodies are actually immersed in a space of four dimensions, but that only three of these present themselves to us as such, the fourth manifesting itself to us as color. We can imagine a being

so organized that differences of color, *i. e.*, differences in the rapidity of ether vibrations, would present themselves to him somewhat as distances from a fixed plane do to us. His intuition would, in a sense, be closer to the truth than ours. For, he would see at a glance that the differences which we attribute to color are relations of a quantitative rather than of a qualitative nature, an idea at which *we* can arrive only as a consequence of elaborate theories and complicated experiments.

Without asserting such to be the case, we cannot dismiss offhand the possibility that absolute space, if it exists at all, has more than three dimensions, that these additional dimensions do not however present themselves to us as such, but become sensible in a masked form as color, electric charges, etc. There is no logical reason why a mathematical theory of color or of electricity might not be built upon such a basis.

Unless we assume dogmatically that space is exactly as it appears to us because we create it with all of its properties, so that objectively it does not exist at all, and unless we deny, dogmatically, the possibility that some other race of thinking beings might possess a space concept also subjective, but different from ours, there remains only one possible way of saving the proposition that space cannot have more than three dimensions, and that And that would be the trivial method of defining it in that way.

ON THE NUMERICAL FACTORS OF CERTAIN ARITHMETIC FORMS.*

By R. D. CARMICHAEL, Princeton University.

The methods used by Dickson in an article on the cyclotomic function† may be applied with some change so as to obtain a set of more general propositions than those at which he arrives and of such nature that his theorems are the simplest cases of those thus obtained. The object of this paper is to generalize the theorems of Dickson; and his method of proof will be freely employed.

We shall let

$$Q_n(x)=0$$

be the equation whose roots are the primitive nth roots of unity without repetition. Then $Q_n(x)$ is a polynomial in x with integral coefficients, that of the highest power being unity. The degree of the equation is $\phi(n)$, $\phi(n)$ being Euler's function.

*Read before the American Mathematical Society, February 29, 1908.

†The American Mathematical Monthly, Vol. XII, pp. 86–89.

As a permanent notation, set

$$n=p_1^{a_1}\, p_2^{a_2}\dots p_r^{a_r}$$

where the p's are different primes and the a's are integers. From the theory of the primitive roots of unity, we have*

$$x^n-1=\Pi\, Q_d(x), \tag{1}$$
$$x^{n/p_1}-1=\Pi\, Q_\delta\,(x), \tag{2}$$

where d ranges over all the divisors of n, and δ over all the divisors of n/p_1. Dividing equation (1) by equation (2), member for member, we have

$$x^{n(p_1-1)/p_1}+x^{n(p_1-2)/p_1}+\dots+x^{n/p_1}+1=\Pi\, Q_c(x), \tag{3}$$

where c ranges over all the divisors of n containing the factor $p_1^{a_1}$. Hence $Q_c(x)$ is a polynomial in x with integral coefficients.

Let β be any integer positive or negative, zero excluded, and α any integer prime to β. (If $\beta=1$, α is any integer whatever.) In (3) replace x by α/β. Multiply both terms of the resulting equation by $\beta^{n(p_1-1)/p_1}$; we have

$$\begin{aligned}\alpha^{n(p_1-1)/p_1}&+\alpha^{n(p_1-2)/p_1}\,\beta^{n/p_1}+\alpha^{n(p_1-3)/p_1}\,\beta^{2n/p_1}+\dots\\&+\alpha^{n/p_1}\,\beta^{n(p_1-2)/p_1}+\beta^{n(p_1-1)/p_1}\\&=\beta^{n(p_1-1)/p_1}\,\Pi\, Q_c(\alpha/\beta).\end{aligned} \tag{4}$$

Denote by $Q_c(\alpha,\beta)$ the quantity $\beta^{\phi(c)}Q_c(\alpha/\beta)$. It is evident that $Q_c(\alpha,\beta)$ is a homogeneous polynomial in α, β with integral coefficients. Equation (4) may now take the following form:

$$\alpha^{n(p_1-1)/p_1}+\alpha^{n(p_1-2)/p_1}\,\beta^{n/p_1}+\dots+\alpha^{n/p_1}\,\beta^{n(p_1-2)/p_1}+\beta^{n(p_1-1)/p_1}=\Pi\, Q_c(\alpha,\beta). \tag{5}$$

In $x^{n/p_1}-1$ replace x by α/β and multiply by β^{n/p_1}. This gives $\alpha^{n/p_1}-\beta^{n/p_1}$. We are now able to find the highest common factor of $\alpha^{n/p_1}-\beta^{n/p_1}$, and $Q_c(\alpha,\beta)$. To this end divide the first member of (5) by $\alpha^{n/p_1}-\beta^{n/p_1}$. A remainder of $p_1.\beta^{n(p_1-1)/p_1}$ is found. This remainder must contain the greatest common divisor sought. But since α and β are relatively prime $\alpha^{n/p_1}-\beta^{n/p_1}$ is not divisible by β. Therefore 1 or p_1 is the greatest common divisor of $\alpha^{n/p_1}-\beta^{n/p_1}$ and $\Pi Q_c(\alpha,\beta)$, the second member of (5). Therefore only one factor of $\Pi Q_c(\alpha,\beta)$ can contain p_1 if $\alpha^{n/p_1}-\beta^{n/p_1}$ contains p_1^2. Hence we have the following theorem:

*See Bachmann's *Kreistheilung*, especially the third and fifth lectures.

THEOREM I. *When α and β are relatively prime integers, the greatest common divisor of $\alpha^{n/p_1}-\beta^{n/p_1}$ and any $Q_c(\alpha, \beta)$ is 1 or p_1, n and c being defined as before. Also, not more than one of the numbers $Q_c(\alpha, \beta)$ contains the factor p_1 when $\alpha^{n/p_1}-\beta^{n/p_1}$ contains p_1^2.*

Suppose that $\alpha^{n/p_1}-\beta^{n/p_1}$ is divisible by p_1, and set this number equal to kp_1, where k is an integer. Then

$$\alpha^{n/p_1}=\beta^{n/p_1}+kp_1.$$

Substituting this value of α^{n/p_1} in the left member of (5) and combining like powers of β, we have

$$p_1\beta^{n(p_1-1)/p_1}+\tfrac{1}{2}p_1(p_1-1)kp_1\beta^{n(p_1-2)/p_1}+\text{terms in } p_1^2,\ p_1^3,\ \ldots$$

Therefore p_1^2 divides the first member of (5) only when

$$p_1\beta^{n(p_1-1)/p_1}+\tfrac{1}{2}p_1(p_1-1)kp_1\beta^{n(p_1-2)/p_1}\equiv 0 \pmod{p_1^2}.$$

Now p_1 is a factor of $\alpha^{n\div p_1}-\beta^{n\div p_1}$, a number to which β is evidently prime; for α and β are relatively prime integers. Hence β is not divisible by p_1. Then the above congruence reduces to

$$p_1\beta^{n/p_1}+\tfrac{1}{2}p_1(p_1-1)kp_1\equiv 0 \pmod{p_1^2}. \tag{6}$$

It is evident that this does not hold when $p_1>2$. Hence the second member of (5) does not contain p_1^2 when $\alpha^{n\div p_1}-\beta^{n\div p_1}$contains $p_1>2$. Hence

THEOREM II. *When α and β are relatively prime integers and p_1 is an odd prime dividing $\alpha^{n\div p_1}-\beta^{n\div p_1}$, not more than one of the numbers $Q_c(\alpha, \beta)$, c as before, is divisible by p_1, and none of them is divisible by p_1^2.*

Consider the case $p_1=2$ in congruence (6) above. We have

$$2\beta^{n\div 2}+2k\equiv 0 \pmod 4,$$

which reduces to

$$\beta^{n\div 2}+k\equiv 0 \pmod 2.$$

Hence β and k are both odd or both even. Now the equation $\alpha^{n\div p_1}-\beta^{n\div p_1}=kp_1$ reduces in the present case to

$$\alpha^{n\div 2}-\beta^{n\div 2}=2k. \tag{7}$$

If k and β are both even, then (7) shows that α is also, contrary to the as-

sumption of α and β relatively prime. Hence k and β are both odd; and therefore by (7) α is also. Moreover it is evident from (7) that of the numbers $\alpha^{n\div 2}$, $\beta^{n\div 2}$ one is congruent to 1 and the other to 3 modulo 4. This requires that $n/2$ shall be odd and that of the numbers α and β one shall be congruent to 1 and the other to 3 modulo 4.

Since $n/2$ is odd it follows that the range of c is over a set of numbers each of which is twice an odd number, unity being included among these odd numbers. u being such an odd number it is evident that $Q_{2u}(\alpha, \beta)$ is equal to or a factor of $\alpha^{u-1}+\alpha^{u-2}\beta+\ldots+\beta^{u-1}$, when $u>1$. But since neither α nor β is even, this last expression contains an odd number of odd terms and is therefore odd. Hence $Q_{2u}(\alpha, \beta)$ is odd when $u>1$. Also, $Q_2(\alpha, \beta)=\alpha+\beta$ and is now divisible by 4; for $\alpha+\beta\equiv 1+3 \pmod 4$. From these considerations we deduce the following proposition:

THEOREM III. *If α and β are relatively prime integers and 2 is a factor of $\alpha^{n/2}-\beta^{n/2}$, then $Q_c(\alpha, \beta)$ is divisible by 2^2 only when $c=2$ and of the numbers α, β one is congruent to* 1 *and the other to* 3 *modulo* 4.

We shall now determine the case in which $Q_c(\alpha, \beta)$ is divisible by p_1.

Set $c=mp_1^{a_1}$, m being an integer >1 and not divisible by p_1. Dickson has shown* that

$$Q_c(x)=Q_n(x^{p_1^{a_1}})\div Q_m(x^{p_1^{a_1-1}}).$$

If in this equation x is replaced by α/β and each function Q is multiplied by that power of β whose index is equal to the degree of the function, there results the following equation:

$$Q_c(\alpha, \beta)=Q_m(\alpha^{p_1^{a_1}}, \beta^{p_1^{a_1}})\div Q_m(\alpha^{p_1^{a_1-1}}, \beta^{p_1^{a_1-1}}). \tag{8}$$

Now by Fermat's Theorem, $\alpha^{p_1}\equiv\alpha$, $\beta^{p_1}\equiv\beta \pmod{p_1}$. Therefore the second member of (8) is congruent to

$$Q_m(\alpha, \beta)\div Q_m(\alpha, \beta)$$

modulo p_1. Now this quantity is 1 unless $Q_m(\alpha, \beta)\equiv 0 \pmod{p_1}$. Hence $Q_c(\alpha, \beta)\equiv 1 \pmod{p_1}$ unless $Q_m(\alpha, \beta)\equiv 0 \pmod{p_1}$.

Consider the case when $Q_m(\alpha, \beta)\equiv 0 \pmod{p_1}$. Now $Q_m(\alpha, \beta)$ divides algebraically

$$(\alpha^m-\beta^m)\div(\alpha^{m/q}-\beta^{m/q})=\alpha^{m(q-1)/q}+\alpha^{m(q-2)/q}\beta^{m/q}+\ldots+\beta^{m(q-1)/q}, \tag{9}$$

where q is any prime factor of m. If

*THE AMERICAN MATHEMATICAL MONTHLY, Vol. XII, p. 86.

(10) $$\alpha^{m/q} \equiv \beta^{m/q} \pmod{p_1},$$

the second member of (9) is congruent to $q\,\beta^{m(q-1)/q} \pmod{p_1}$. Then there is some integer k such that

$$kQ_m(\alpha, \beta) \equiv q\,\beta^{m(q-1)/q} \pmod{p_1}.$$

Now q is different from p_1; for it is a factor of m which is prime to p_1. Therefore when (10) holds $Q_m(\alpha, \beta)$ is not divisible by p_1 when β is not so divisible. But if β is divisible by p_1, $Q_m(\alpha, \beta)$ is not congruent to zero modulo p_1; and hence $Q_c(\alpha, \beta) \equiv 1 \pmod{p_1}$.

There is left the case $\alpha^{m/q}$ not congruent to $\beta^{m/q}$ and $Q_m(\alpha, \beta) \equiv 0$ modulo p_1. From the last congruence follows

(11) $$\alpha^m - \beta^m \equiv 0 \pmod{p_1},$$

for $Q_m(\alpha, \beta)$ is a factor of $\alpha^m - \beta^m$. But since

$$\alpha^{m/q} - \beta^{m/q} \text{ is not} \equiv 0 \pmod{p_1},$$

where q is any prime divisor of m, it follows that m is the least exponent for which congruence (11) holds.

These considerations lead to the following theorem:

THEOREM IV. *If α and β are relatively prime integers and $c = mp_1^{a_1}$, where $m > 1$ and not divisible by p_1, then $Q_c(\alpha, \beta)$ is divisible by p_1 if and only if the congruence $\alpha^m \equiv \beta^m \pmod{p_1}$ holds for m and for no exponent less than m. In all other other cases $Q_c(\alpha, \beta) \equiv 1 \pmod{p_1}$.*

If m contains any prime factor greater than p_1, the condition that $\alpha^m \equiv \beta^m \pmod{p_1}$ holds for no exponent less than m is not satisfied; and therefore we have the following corollary:

COROLLARY I. *No one of the prime factors of c except the greatest can divide $Q_c(\alpha, \beta)$. For other prime factors of c we have always $Q_c(\alpha, \beta) \equiv 1$.*

COROLLARY II. *When c contains an odd factor $Q_c(\alpha, \beta)$ is odd, α and β being relatively prime integers.*

If we set $v = p_1^{a_1}$, we may write

$$Q_v(\alpha, \beta) = (\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}}) \div (\alpha^{p_1^{a_1-1}} - \beta^{p_1^{a_1-1}}) \equiv \frac{\alpha-\beta}{\alpha-\beta} \pmod{p_1}.$$

Hence $Q_v(\alpha, \beta) \equiv 1 \pmod{p_1}$ unless $\alpha \equiv \beta \pmod{p_1}$. In the latter case it is evident that $Q_v(\alpha, \beta) \equiv 0 \pmod{p_1}$. Therefore

THEOREM V. *If α and β are relatively prime integers and $v = p_1^{a_1}$,*

$Q_v(\alpha, \beta) \equiv 0$ *or* 1 (mod p_1) *according as* $\alpha - \beta$ *is or is not congruent to* 0 (*mod* p_1).

COROLLARY I. *If* $\alpha \equiv \beta$ (*mod* p_1), $\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}}$ *is divisible bg* $p_1^{a_1+1}$.

For,

$$\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}} = \Pi\, Q_d(\alpha, \beta),$$

where d runs over all the a_1+1 divisors of $p_1^{a_1}$. Each factor of the second member contains p_1.

If p_1 is an odd prime the preceding corollary can be made more exact in view of the last clause in Theorem II:

COROLLARY II. *If* p_1^r *is the highest power of an odd prime* p_1 *contained in* $\alpha - \beta$, *then* $\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}}$ *is divisible by* $p_1^{a_1+r}$ *but by no higher power of* p_1.

Combining Corollary II with Theorem IV we readily deduce the following.

THEOREM VI. *If* α *and* β *are relatively prime integers and* $\alpha - \beta$ *contains* p_1^r *and no higher power of* p_1, *then* $\alpha^{mp_1^{a_1}} - \beta^{mp_1^{a_1}}$ *contains* $p_1^{a_1+r}$ *and no higher power of* p_1 *m as before being prime to* p_1.

We are now prepared to prove the following theorem:

THEOREM VII. *If* α *and* β *are relatively prime integers,* $Q_c(\alpha, \beta)$ *has a prime factor not dividing* $\alpha^s - \beta^s$ $(s<c)$, *except in the cases*

1) $c=2$, $\beta=1$, $\alpha=2^k-1$, *where k is an integer.*

2) $Q_c(\alpha, \beta)=p$, *the greatest prime factor of c, and* $\alpha^{n/p} - \beta^{n/p} \equiv 0$ (mod p).

3) $Q_c(\alpha, \beta)=1$.*

Every prime factor of $Q_c(\alpha, \beta)$ is evidently contained in $\alpha^c - \beta^c$. Now if $\alpha^s - \beta^s$ and $\alpha^c - \beta^c$ contain a common prime factor q, it follows from the congruences

$$\alpha^s \equiv \beta^s, \quad \alpha^c \equiv \beta^c \pmod q$$

that

$$\alpha^l \equiv \beta^l \pmod q,$$

where l is the greatest common divisor of s and c. Hence every common prime factor of $\alpha^s - \beta^s$ and $Q_c(\alpha, \beta)$ is contained in some $\alpha^d - \beta^d$ where d is a divisor of n. Now evidently, every $\alpha^d - \beta^d$ is contained in $\alpha^{c/p_i} - \beta^{c/p_i}$ where for p_i is put the different prime factors of n. By Corollary I to Theorem IV, $Q_c(\alpha, \beta)$ does not contain p unless p is the greatest prime factor of c. Then if $\alpha^{c/p} - \beta^{c/p}$ and $Q_c(\alpha, \beta)$ contain the common factor p, $Q_c(\alpha, \beta)$ does not contain p^2 unless $n=2$, as is seen from Theorems II and III. If $c=2$, $Q_c(\alpha, \beta)= \alpha+\beta$, and this contains a factor different from those of $\alpha - \beta$ unless $\beta=1$, $\alpha= 2^k-1$. This accounts for the first exception.

If $n \neq 2$ and $Q_c(\alpha, \beta)$ and $\alpha^{c/p} - \beta^{c/p}$ contain the common factor p,

*The author knows no numerical case for positive β under exception 3) and only one under exception 2); namely, $Q_6(2, 1)=2^2-2+1=3$.

$Q_c(\alpha, \beta)$ does not contain p^2, as we have just seen. But these two numbers contain no other factor in common, as is seen from Theorem I. Hence $Q_c(\alpha, \beta)$ contains a prime not in $\alpha^{c/p}-\beta^{c/p}$ at least in every case for which $Q_c(\alpha, \beta)\neq p$ and also in every case for which $\alpha^{c/p}-\beta^{c/p}$ is not divisible by p, unless $Q_c(\alpha, \beta)=1$.

COROLLARY. *$\alpha^c-\beta^c$ has always a prime factor not dividing $\alpha^s-\beta^s$ $(s<c)$ except in the cases mentioned in Theorem VII.*

NOTE ON THE EXTENSION OF THE EXPONENTIAL THEOREM.

By E. D. ROE, JR., Syracuse University.

In the writer's paper in the June-July number of the MONTHLY, pp. 101-106, it will be observed that the complex roots, infinite in number, arising from the application of an incommensurable exponent are tacitly neglected, only the single real root being used. In fact this is the only root that is of much practical value. It is the only root that is usually considered in the extension of the binomial theorem for the expansion of $(1+x)^n$, even for a commensurable fractional exponent. The finite number of complex roots can be easily expressed if wanted. But when the exponent is incommensurable the number of complex roots becomes infinite and the complex roots become indeterminate.

In both cases, vis., of the binomial theorem, and the exponential theorem, which depends on the binomial theorem, the developments already obtained would have to be multiplied by $\cos\varphi+i\sin\varphi$, to obtain the complete development, where φ admits the value of zero, and has besides an infinite number of values which are indeterminate, when the exponent is incommensurable. All the values would have the same modulus.

These facts are so obvious that it will doubtless appear superfluous to many readers to call attention to them; yet for others it may not be amiss to do so.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

319. Proposed by C. N. SCHMALL, New York City.

A man desires to purchase eggs at 5 cents, 1 cent, and $\frac{1}{2}$ cent, respectively, in such numbers that he will obtain 100 eggs for a dollar. How many solutions in rational integers?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; J. W. CLAWSON, Ursinus College, Collegeville, Pa.; V. M. SPUNAR, Pittsburg, Pa.; and H. C. FEEMSTER, York College, York, Neb.

Let x=number at 5 cents, y=number at 1 cent, z=number at $\frac{1}{2}$ cent.
Then $x+y+z=100=5x+y+\frac{1}{2}z$... (1, 2).
Eliminating z we get the indeterminate equation, $9x+y=100$.
$\therefore y=100-9x$.
This equation gives us eleven integral solutions, as follows:
x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
y=91, 82, 73, 64, 55, 46, 37, 28, 19, 10, 1.
z=8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88.

Also solved by S. F. Norris, S. A. Corey, B. Kramer, J. Scheffer, T. J. Fitzpatrick, and Theodore L. DeLand.

II. Solution by PROFESSOR S. F. NORRIS, Baltimore City College, Baltimore, Md., and J. E. SANDERS, Weather Bureau, Chicago, Ill.

Average price=1												
	5	1	2	3	4	5	6	7	8	9	10	11
	1	91	82	73	64	55	46	37	28	19	10	1
	$\frac{1}{2}$	8	16	24	32	40	48	56	64	72	80	88

NOTE. This is the method used in many arithmetics. For the process of reasoning see *e. g.*, Ray's *New Higher Arithmetic*, p. 333, subject, Alligation. ED. F.

320. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Solve for t, $\cos t=m\cos 2t$.

Solution by T. J. FITZPATRICK, Lamoni, Iowa; S. A. COREY, Hiteman, Iowa; J. E. SANDERS, Weather Bureau, Chicago, Ill.; and B. KRAMER, E. M., N. S., Pittsburg, Pa.

$$\cos t=m\cos 2t=m(\cos^2 t-\sin^2 t)=m(\cos^2 t-1+\cos^2)=2m\cos^2 t-m.$$

$$2m\cos^2 t-\cos t=m.$$

$$\cos^2 t-\frac{1}{2m}\cos t=\tfrac{1}{2}.$$

$$\cos^2 t-\frac{1}{2m}\cos t+\frac{1}{16m^2}=\tfrac{1}{2}+\frac{1}{16m^2}=\frac{8m^2+1}{16m^2}.$$

$$\cos t-\frac{1}{4m}=\pm\frac{1}{4m}\sqrt{[(8m^2+1)]}.$$

$$\cos t=\frac{1}{4m}\pm\frac{1}{4m}\sqrt{[(8m^2+1)]}=\frac{1}{4m}\{1\pm\sqrt{[(8m^2+1)]}\}.$$

$$\therefore t=\cos^{-1}\{\frac{1}{4m}[1\pm\sqrt{(8m^2+1)}]\}.$$

Also solved by G. B. M. Zerr, and H. C. Feemster.

GEOMETRY.

344. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

A tinsmith has a sheet of copper in the form of a rectangle, sides a and b. He desires to cut this into two pieces which will form a square when placed together. How can he do this?

I. Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Assuming the base b of the rectangle to be divided into n and the other side a into $n-1$ equal parts. Then, cutting the whole figure by the broken line m, n, m', n', m'', n'', ..., pushing the left part one step down and shove it one step to the right, we will have for a square,

$$(n-1)x=ny.$$

Also, $a=(n-1)x$, and $b=ny$.

$$\therefore b:a=\left(\frac{n}{n-1}\right)^2.$$

As n must be a positive integer, the relation $b:a$ is *restricted* by the last equation. It is satisfied by the following series of numerical values.

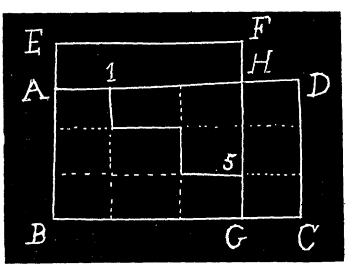
For n=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ..., ∞.

$b:a=\infty$, 4, $2\frac{1}{4}$, $1\frac{7}{9}$, $1\frac{9}{16}$, $1\frac{11}{25}$, $1\frac{13}{36}$, $1\frac{15}{49}$, $1\frac{17}{64}$, $1\frac{19}{81}$, ..., 1.

Hence, the required division is possible when the sides of the rectangle are in the ratio 4 : 1.

Also solvee by S. Lefsehetz.

II. Solution by C. N. SCHMALL, 604 East 5th Street, New York.



Let $ABCD$ be the given rectangular sheet of copper; $AB=a$, $BC=b$. Let us suppose a square (of paper) $EBGF$ to be constructed by the usual method of the mean-proportional. Then the rectangles $AHFE$ and $GCDH$ are equivalent, and their sides are therefore reciprocally proportional.

$\therefore AH:HD=GH:HF$, or $BG:GC=AB:AE$,

$$\text{or } \frac{BG}{GC} = \frac{AB}{AE} \ldots (1).$$

Hence, if BG be a multiple of GC then AB will be a multiple of AE. In that case divide BG by GC and AB by AE. Now cut along the line $A1$, 2 3, 4 5 to G and then bring 2 3, into coincidence with $A1$, and GC into coincidence with 4 5. Then the given rectangle has been transformed into a square. This problem is impossible if the ratios in equation (1) are not integers. In other words, if $AB=a$, and $BC=b$, $EB=\sqrt{(ab)}$, $EA=\sqrt{(ab)}-a$, $GC=b-\sqrt{(ab)}$, the equation (1) becomes

$$\frac{\sqrt{(ab)}}{b-\sqrt{(ab)}} = \frac{a}{\sqrt{(ab)}-a} = \text{an integer} \ldots (2),$$

and the desired section cannot be performed unless (2) be true.

Assume $\dfrac{a}{\sqrt{(ab)}-a} = \dfrac{\sqrt{a}}{\sqrt{(b)}-\sqrt{(a)}} = i$, an integer.

Then $\sqrt{(b)} = \sqrt{(a)}\,(1+\dfrac{1}{i})$, which is satisfied when $i=1$ only.

345. Proposed by LLOYD HOLSINGER, Bradley Polytechnic Institute, Peoria, Ill.

If a variable polygon move in such a way that its n sides turn severally round n fixed points O_1, O_2, ..., O_n while $n-1$ of its vertices slide, respectively, along $n-1$ fixed straight lines v_1, v_2, ..., v_{n-1}, then the last vertex will describe a conic; and the locus of the point of intersection of any pair of non-adjacent sides will also be a conic. Cremona's *Projective Geometry.*

No solution of this problem has been received.

346. Proposed by G. I. HOPKINS, M. A., Professor of Mathematics and Astronomy, High School, Manchester, N. H.

Prove the theorem for finding the lateral area of a frustum of a cone without the use of the theory of limits.

I. Solution by the PROPOSER.

The theorem relating to the lateral area of the frustum of a cone is given in most text books on Solid Geometry, and proved by the theorem of limits. The accompanying proof does not use limits. The lateral area of the frustum of a cone is equal to its slant height multiplied by one half the sum of the circumferences of its bases.

Let $BADE$ be the frustum of the cone OAD. Designate the radii of its bases by R and r, their circumferences by C and c. Also designate the lateral areas of the two cones by L and l, and their slant heights by S and s.

Then $S-s$ represents the slant height of the frustum, and $L-l$ its lateral area.

$L=\frac{CS}{2}$ and $l=\frac{cs}{2}$. $\therefore L-l=\frac{CS-cs}{2}$.

$\frac{S}{s}=\frac{R}{r}=\frac{C}{c}$. $\therefore Sc=Cs$.

$\therefore Sc-Cs=0$. $CS-cs=CS-cs$.

$\therefore CS+Sc-Cs-cs=CS-cs$. $\therefore (C+c)(S-s)=CS-cs$.

$\therefore L-l=\frac{(S-s)(C+c)}{2}$. Q. E. D.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Md.**

Complete the frustum to a full cone, and by cutting along an element spread out the surface into a plane, which can be done with all developable surfaces. The circular arc AB is equal to the circumference of one of the base circles of the frustum$=2\pi R$, and arc $CD=2\pi r$.

$AC=BD=$slant height l of frustum.

From $OA:OC=AB:CD=R:r$, and $OA-OC=l$, we get $OA=\frac{Rl}{R-r}$, $OC=\frac{rl}{R-r}$.

Area of $ABCD=\frac{1}{2}AB.OA-\frac{1}{2}CD.OC=\frac{1}{2}.2\pi R.\frac{Rl}{R-r}-\frac{1}{2}.2\pi r.\frac{rl}{R-r}=\pi l.\frac{R^2-r^2}{R-r}=\pi l(R+r)$.

CALCULUS.

277. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

Find $\frac{d^2x}{ds^2}$ and $\frac{d^2y}{ds^2}$ for $y=c\sinh\frac{x}{c}$.

Solution by J, W. CLAWSON, Collegeville, Pa.

$y=c\sinh\frac{x}{c}$. $\therefore \frac{dy}{dx}=\cosh\frac{x}{c}$. But $ds^2=dx^2+dy^2$.

$$\therefore \left(\frac{ds}{dx}\right)^2=1+\cosh^2\frac{x}{c} \text{ and } \left(\frac{ds}{dy}\right)^2=1+\frac{1}{\cosh^2(x/c)}.$$

$$\therefore \frac{dx}{ds}=(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}} \text{ and } \frac{dy}{ds}=(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}.\cosh\frac{x}{c}.$$

$$\therefore \frac{d^2x}{ds^2}=-\tfrac{1}{2}(1+\cosh^2\frac{x}{c})^{-\frac{3}{2}}.\,2\cosh\frac{x}{c}\sinh\frac{x}{c}.\frac{1}{c}.(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}$$

$$=-\frac{1}{c}\frac{\sinh(x/c)\ \cosh(x/c)}{[1+\cosh^2(x/c)]^2}.$$

$$\frac{d^2y}{ds^2}=-\tfrac{1}{2}(1+\cosh^2\frac{x}{c})^{-\frac{3}{2}}.2\cosh\frac{x}{c}\sinh\frac{x}{c}.\frac{1}{c}.\ (1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}.\cosh\frac{x}{c}$$

$$+\sinh\frac{x}{c}.\frac{1}{c}(1+\cos^2\frac{x}{c})^{-\frac{1}{2}}.(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}=\frac{1}{c}\frac{\sinh(x/c)}{[1+\cosh^2(x/c)]^2}.$$

Also solved by H. C. Feemster, G. B. M. Zerr, V. M. Spunar, and J. Scheffer.

278. Proposed by S. A. COREY, Hiteman, Iowa.

If C be Euler's constant, .577,215,664,9... and if B_1, B_2, B_3, etc., be Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc., prove that

$$C=\tfrac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-\frac{B_4}{8}+\ldots-(1)^m\frac{B_m}{2m}+\ldots.$$

I. Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Euler's constant C may be presented under various forms from among which an elementary one may be the following:

$$C=\lim_{n\doteq\infty}[1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\frac{1}{n}-\log n]=.577,215,664,9\ldots$$

By Taylor's Theorem, we have

$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots+\frac{h^n}{n!}f^{(n)}(a)+\ldots$$

Change a successively into $a+h$, $a+2h$, ..., $a+(n-1)h$ and add, then if we put x for $(a+nh)$, we obtain the following result:

$$f(x)-f(a)=h\,\Sigma f'(x)+\frac{h^2}{2!}\Sigma f''(x)+\frac{h^3}{3!}\Sigma f'''(x)+\ldots+\frac{h^3}{n!}\Sigma f^{(n)}(x)+\ldots$$

where $\Sigma f'(x)=f'(x)+f'(a+h)+\ldots$, $\Sigma f''(x)=f''(a)+f''(a+h)+\ldots$, etc.

Let $\phi(x)=f'(x)$. Then

$$\int_a^{a+nh}\phi(x)dx=h\,\Sigma\phi(x)+\frac{h^2}{2!}\Sigma\phi'(x)+\frac{h^3}{3!}\Sigma\phi''(x)+\ldots+\frac{h^n}{n!}\Sigma\phi^{(n-1)}(x)+\ldots.$$

$$\therefore \Sigma\,\phi\,(x)=\frac{1}{h}\int_a^{a+nh}\phi(x)\,dx-\frac{h}{2\,!}\,\Sigma\,\phi'(x)\;-\frac{h^2}{3\,!}\,\Sigma\,\phi''(x)-\ldots$$

$$-\frac{h^n}{(n+1)\,!}\,\Sigma\,\phi^{(n)}(x)-\ldots(1).$$

Similarly, $$\Sigma\,\phi'(x)=\frac{1}{h}[\phi(x)-\phi(a)]-\frac{h}{2\,!}\,\Sigma\,\phi''(x)-\frac{h^2}{3\,!}\,\Sigma\,\phi'''(x)-\ldots$$

$$-\frac{h^n}{(n+1)\,!}\phi^{(n+1)}(x)-\ldots(2).$$

Now from the series (1) we may eliminate $\Sigma\,\phi'(x)$, $\Sigma\,\phi''(x)$, ..., by the aid of (2), (3), ... In order to do it multiply (2) by A_0h, (3) by A_1h^2, ..., then add the results and determine A_0, A_1, A_2, ..., by equations

$$A_0+\frac{1}{2\,!}=0.$$

$$A_1+\frac{A_0}{2\,!}+\frac{1}{3\,!}=0.$$

$$A_2+\frac{A_1}{2\,!}+\frac{A_0}{3\,!}+\frac{1}{4\,!}=0.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\therefore \Sigma\,\phi(x)\;=\frac{1}{h}\int_a^{a+nh}\phi(x)\,dx+A_0[\phi(x)-\phi(a)]+A_1[\phi'(x)-\phi'(a)]h$$

$$+A_2[\phi''(x)-\phi''(a)]h^2+\ldots$$

As it has been shown that $\Sigma\,\phi(x)$ can be put in this form, where A_0, A_1, A_2, ..., A_n, ... are numerical quantities independent of the variable x and of the function $\phi(x)$, viz:

$$\Sigma\,\phi(x)\;=\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{B_1}{2\,!}x^2\;-\frac{B_2}{4\,!}x^4+\ldots\;(-1)^{m+1}\frac{B_m}{2m\,!}+\ldots,\text{ where}$$

$A_m=(-1)^{m+1}\dfrac{B_m}{2m\,!}$, and B_1, B_2, B_3, ..., B_m are the well known Bernoulli's numbers ($A_0=-\frac{1}{2}$).

$$\therefore \Sigma\,\phi(x)=\frac{1}{h}\int_a^x\phi(x)dx-\tfrac{1}{2}[\phi(x)-\phi(a)]+\frac{B_1}{2\,!}[\phi'(x)-\phi'(a)]h$$

$$-\frac{B_2}{4\,!}[\phi'''(x)-\phi'''(a)]h^3+\ldots$$

Also,

$$\therefore \Sigma\,\phi(x)=C+\frac{1}{h}\int\phi(x)dx-\tfrac{1}{2}\phi(x)+\frac{B_1}{2}\phi'(x)\frac{h}{1\,!}-\frac{B_2}{4}\phi'''(x)\frac{h^3}{3\,!}+\ldots$$

$$\ldots(-1)^{m+1}\frac{B_m}{2_m}\phi^{(2m-1)}(x)\frac{h^{2m-1}}{(2m-1)\,!}+\ldots(M),$$

where C represents a series of terms independent of x.

Taking $\phi(x)=1/x$ and $h=1$ then we get, adding $(1/x)$ to both sides,

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{x}=C+\log x+\frac{1}{2x}-\frac{1}{12x^2}+\ldots$$

Hence, by making $x=\infty$ we infer that in this example C is Euler's constant (according to our definition).

But on the other hand in (M),

$$C=-\frac{1}{h}\int\phi(a)da+\tfrac{1}{2}\phi(a)+\frac{B_1}{2}\phi'(a)\frac{h}{1\,!}-\frac{B_2}{4}\phi'''(a)\frac{h_3}{3\,!}+$$

$$\ldots(-1)^{m+1}\frac{B_m}{2m}\phi^{(2m-1)}(a)\frac{h^{2m-1}}{(2m-1)\,!}\ldots(N).$$

Hence, put $\phi(a)=1/a$, $h=1$ (as above), and then $a=1$.

$$\therefore C\equiv\tfrac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-\ldots(-1)^{m+1}\frac{B_m}{2m}\ldots\quad \text{Q. E. D.}$$

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Taking $a=1$, $x=m$ in the development in problem 237, Calculus, we get

$$\log(1+m)=0+\tfrac{1}{2}\left[\frac{1}{1+m}+1+2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{m}\right)\right]$$

$$+\frac{B_1}{2\,!}\left[\frac{1}{(1+m)^2}-1\right]-\frac{B_2}{4\,!}\left[\frac{1}{(1+m)^4}-3!\right]+\text{etc.}$$

Let m approach ∞. Then

$$\lim_{n \doteq \infty} \log n = \left[\tfrac{1}{2} + \sum_{p=2}^{p=\infty} \frac{1}{p}\right] - \left[\frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \dots\right]$$

But $\lim_{n \doteq \infty} \log n - \sum_{p=2}^{p=\infty} \frac{1}{p} = -C+1.$

$$\therefore -C+1 = \tfrac{1}{2} - \frac{B_1}{2} + \frac{B_2}{4} - \frac{B_3}{6} + \frac{B_4}{8} - \dots$$

$$\therefore C = \tfrac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \frac{B_4}{8} + \dots.$$

Also solved by the Proposer.

MECHANICS.

227. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

II. Solution by the PROPOSER.

Let AB be the tunnel; C its mid-point; D the position of the sphere at any time t; O the center of the earth; $OD=y$, $AC=a$, $DC=x$, $\angle DOC=\theta$; f' the acceleration at D. Then $f' : y = g : R$ or $f' = gy/R$. But $y = x/\sin\theta$; also $f = f' \sin\theta$. $\therefore f = gx/R$.

$$\text{Now} \int_{v_1}^{v} v\,dv = f\int_{x}^{a} dx = \frac{g}{R}\int_{x}^{a} x\,dx.$$

$$\therefore v^2 = (dx/dt)^2 = g/R(a^2-x^2) + v_1^2.$$

$$\therefore t = \sqrt{(R/g)}\int_0^a \frac{dx}{\sqrt{[Rv_1^2 + g(a^2-x^2)]}}.$$

$$\therefore t = (R/g)\sin^{-1}\frac{a\sqrt{g}}{\sqrt{[Rv_1^2+ga^2]}}.$$

If the sphere starts from rest, $v_1=0$.

$\therefore t = (\pi/2)\sqrt{(R/g)}$.

If the tunnel is perfectly rough and F is the friction, we get for the equations of motion: $d^2x/dt^2 + gx/R + F = 0$; $k^2(d^2\phi/dt^2) = bF$; $bd\phi = dx$.

$$\therefore \frac{d^2x}{dt^2}\left(\frac{b^2+k^2}{b^2}\right)+\frac{gx}{R}=0. \quad \therefore \frac{dx^2}{dt^2}+\frac{5gx}{7R}=0.$$

$$\left(\frac{dx}{dt}\right)^2=\frac{5g}{7R}(a^2-x)^2 \text{ or } T=\frac{\pi}{2}\sqrt{\frac{7R}{5g}}.$$

Let R=20902410 feet, g=32.10614 feet. Then $\sqrt{(R/g)}$=806.871.
t=1267.433 seconds=21 minutes, 7.433 seconds.
T=1499.627 seconds=24 minutes, 59.627 seconds.

t, T the time for the sphere to move from rest at the surface to the middle of the tunnel is constant for tunnels of all lengths and the same as for a tunnel passing through the center of the earth.

228. Proposed by J. E. ROSE, Mount Angel College, Mount Angel, Oregon.

AB, BC are two uniform rods freely hinged at B, whose weights are W, $4W$, and lengths $2a$, $4a$, respectively. The ends A, C of the rods are attached to small rings which slide on a rough horizontal wire. When the distance between the rings is the greatest for which equilibrium can exist, both of them are on the point of slipping. Find the coefficient of friction.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and S. LEFSEHETZ, Pittsburg, Pa.

Let R, R' be the reactions at C, A, respectively; μR, $\nu R'$ the friction at these points; $\angle BAC=\theta$, $\angle BCA=\phi$.

Then $R+R'=5W$, $\mu R=\nu R'$... (1, 2).

Taking moments around B we get

$2R'\cos\theta=2\nu R'\sin\theta+W\cos\theta$ or $R'=W/[2(1-\nu\tan\theta)]$... (3).
$2R\cos\phi=2\mu R\sin\phi+4W\cos\phi$ or $R=2W/(1-\mu\tan\phi)$... (4).

(1, 2) in (3) and (4) gives $9\mu-\nu=10\mu\nu\tan\theta$, $3\nu-2\mu=5\mu\nu\tan\phi$.

$$\therefore \frac{5}{\tan\phi+6\tan\phi}=\nu, \quad \frac{5}{4\tan\theta+9\tan\phi}=\mu.$$

In this problem, $\mu=\nu$. $\therefore \mu=\frac{4}{5}\cot\theta=\frac{1}{5}\cot\phi$. $\therefore \tan\theta=4\tan\phi$.
Also $\sin\theta : \sin\phi=2:1$, or $\sin\theta=2\sin\phi$.
$\therefore \tan\phi=\frac{1}{2}$. $\therefore \mu=\frac{2}{5}$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

324. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the *finite* series

$$\frac{16n^2-2^2}{4!}-\frac{(16n^2-2^2)(16n^2-4^2)}{6!}+\frac{(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}-\cdots$$

where n is a positive integer.

325. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

I have a chronometer whose rate is uniform. When it indicates t_1 time at Washington I find that it is h_1 hours slow. I take it to Philadelphia and when it indicates t_2 time, the local time of Philadelphia is h_2 hours faster. I bring my chronometer back to Washington and find that when it indicates t_3 time it is h_3 hours slow. If $t_1=5$ A. M., $t_2=7$ hours, 54 minutes, $t_3=11$ hours, 46 minutes A. M., $h_1=1$ hour, $h_2=1$ 203/900 hours, $h_3=1$ 7/30 hours, find the difference of longitude between Washington and Philadelphia.

GEOMETRY.

351. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Given an isosceles right triangle with hypothenuse h; an isosceles triangle with two sides h and two angles $A=22°\ 30'$; a right angle triangle with the same angle A and opposite side $h/\sqrt{2}$; a triangle with the same angle A, opposite side h, and an angle 45°. Form a triangle whose four pieces are these four triangles, and prove geometrically that it is isosceles.

352. Proposed by G. I. HOPKINS, Professor of Astronomy, High School, Manchester, N. H.

Required, to construct the triangle, having given the base, vertical angle and sum of the altitude and the two remaining sides.

353. Propoaed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

In a given circle place two chords which shall be in a given ratio and also a given distance apart.

CALCULUS.

282. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

A rectangular beam of length l and width w is taken horizontally from a hall of width b into a corridor at right angles to the hall. Find the width of the smallest corridor into which it can be taken.

283. Proposed by B. F. FINKEL, Ph. D., Drury College.

By means of the calculus, determine the angle of minimum deviation of a ray of monochromatic light in passing through a triangular prism.

MECHANICS.

236. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at the ends, is loaded with c pounds per running foot at the ends and increases uniformly to the center, where it is b pounds per running foot. Find deflection at center due to this load.

237. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

In a naval action an officer observes that in the case of two guns firing, at elevations α and β, respectively, the projectile of the former falls a feet short of the target while that of the latter lands b feet beyond. The initial velocity being the same in both cases, prove that the *true* elevation is

$$\tfrac{1}{2}\sin^{-1}\left[\frac{a\sin2\beta+b\sin2\alpha}{a+b}\right].$$

(Suggested by problem 29, page 219, Jeans' *Theoretical Mechanics.*)

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

168. Proposed by A. H. HOLMES, Brunswick Maine.

Find integral values for x, y, u and v from the following:

$$uv-xy=25x+29y+29u+29v-112.$$
$$3v-5u+5y-x=102.$$
$$4y-3v=419.$$

169. Proposed by R. D. CARMICHAEL, Princeton University.

Let $Q_n(x)=0$ be the equation whose roots are all the primitive nth roots of unity without repetition. In $Q_n(x)=0$ replace x by α/β, a fraction in its lowest terms, and clear of fractions. Let $Q_n(\alpha, \beta)$ represent the resulting first member. Set $n=mp$ where p is the largest prime factor of n. It is required to find all the integral values of α, β, m, p satisfying the following relations:

$$(1)\quad Q_{mp}(\alpha, \beta)=p,$$
$$(2)\quad \alpha^m-\beta^m\equiv0 \pmod{p}.$$

One such solution is: $\alpha=2$, $\beta=1$, $m=2$, $p=3$. (See MONTHLY, Vol. XII, p. 89.)

NOTES AND NEWS.

The next annual meeting of the American Association for the Advancement of Science will be held at Boston during the week beginning December 27. Titles and abstracts of papers intended for the section of Mathematics and Astronomy should reach Professor G. A. Miller, 907 West Nevada Street, Urbana, Ill., before December 15.

Professor G. H. Scott of Yankton College, S. D., is spending the present year at the University of Illinois as assistant in mathematics, and graduate student. M.

The American Mathematical Society decided to contribute 5000 francs towards the publication of the complete works of the noted Swiss mathematician, Leonard Euler. M

The following promotions and appointments for the present year to positions in Mathematics in colleges and universities will be of interest to the readers of the MONTHLY:

R. D. Carmichael to a teaching fellowship at Princeton University.

Dr. N. J. Lennes to an instructorship in the Massachusetts Institute of Technology.

Dr. F. L. Griffin to an assistant professorship at Williams College.

G. P. Paine, of Ripon College, to an assistant professorship at the Univessity of Minnesota.

Dr. W. R. Longley to an assistant professorship at Yale University.

Dr. C. N. Haskins to an associate professorship at Dartmouth College.

T. Hildebrandt to an instructorship at the University of Michigan.

Dr. A. L. Underhill to an assistant professorship at the University of Minnesota.

L. L. Silverman to an instructorship at Cornell University.

Dr. G. D. Birkhoff to an assistant professorship at Princeton University.

E. J. Moulton to an instructorship at the University of Wisconsin.

Dr. Thomas Buck to an instructorship at the University of Illinois.

Dr. Arnold Dresden to an instructorship at the University of Wisconsin.

Dr. C. L. E. Moore to an assistant professorship at Massachusetts Institute of Technology.

Dr. H. E. Buchanan to an instructorship at the University of Wisconsin.

Dr. J. H. Maclagan-Wedderburn to a preceptorship at Princeton University. S.

The American Federation of Teachers of the Mathematical and the Natural Sciences, at its meeting in Baltimore, December, 1908, authorized the appointment by its council of a committee of fifteen on a syllabus of geometry for secondary schools. A similar committee had been authorized by the Mathematics Section of the Secondary Department of the National Education Association at its meeting in Cleveland in 1908. It has been dedided that the same committee shall act under the joint auspices of the two national bodies. The membership of the committee is as follows: H. E. Slaught, chairman, Univeasity of Chicago: C. L. Bouton, Harvard Univer-

sity; Florian Cajori, Colorado College; H. E. Hawks, Yale University; E. R. Hardrick, University of Missouri; H. L. Rietz, University of Illinois; D. E. Smith, Teachers College, Columbia University; William Betz, East High School, Rochester, N. Y.; E. L. Brown, North High School, Denver, Colo.; W. B. Carpenter, Mechanics Arts High Suhool, Boston, Mass.; W. W. Hart, Shortridge High School, Indianapolis, Ind.; F. K. Newton, Phillips Andover Academy, Andover, Mass.; E. R. Smith, Brooklyn, N. Y. Polytechnic School; R. L. Short, Technical High School, Cleveland, O.; and Mabel Sykes, South Chicago High School, Chicago, Ill.

The committee is already at work, having resolved itself into three sub-committees of five members each, under the chairmanship of Professors D. E. Smith, E. R. Hedrick, and H. L. Rietz, respectively, for the study of the three phases: Logical considerations, lists of basal theorems, and exercises and applications. S.

Among the important mathematical meetings in the near future are the following:

The Central Association of Science and Mathematics Teachers at the University of Chicago, November 26, 27, 1909.

The American Federation of Teachers of the Mathematical and the Natural Sciences at Boston during the holidays.

The American Mathematical Society, in connection with Section A of the American Association for the Advancement of Science, at Boston during the holidays.

The Chicago Section of the American Mathematical Society at the University of Chicago December 31, 1909, January 1, 1910,

The Southwestern Section of the American Mathematical Society at the University of Missouri in the Thanksgiving recess.

B. F. Finkel will address the Southwest Section of the Missouri State Teachers' Association on November 26, on "The Teaching of Mathematics for College Entrance."

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THE TEACHING OF MATHEMATICS IN THE COLLEGES.

By H. E. SLAUGHT, The University of Chicago.

The first decade of the twentieth century is witnessing a wide spread wave of interest in the better teaching of secondary mathematics. This interest includes both the subject-matter and the form of presentation and arrangement in the curriculum and also the preparation of teachers for secondary schools.

Among the important activities pertaining to the subject-matter and its presentation may be mentioned the large number of associations of teachers, banded together for the purpose of improving the teaching of mathematics, in cities, in states, and in large sections of the country, such as the New England Association, the Middle States and Maryland Association, and the Central Association. In all of these associations, by papers, discussions, syllabi, committee reports, and experience meetings, every phase of secondary mathematical work has undergone the most careful reconsideration.

With respect to the preparation of teachers of secondary subjects, a great change has also taken place in recent years. Whereas, formerly there was no prevalent demand for college graduates as high school teachers, it is now true that it is very difficult for a teacher without a college degree to secure an appointment in any first-class high school. But further than this, specific training in the principles and practice of education is coming to be demanded. A notable instance of this is the state requirement in California, where no teacher can be certificated without a year's work in education in the University of California or in some institution where work of a high grade is offered.

While doubtless this standard of preparation has been set by the better equipped universities and by the few schools or colleges of education, yet the demand for such training is now being reflected back from the schools and compelling colleges and universities all over the country to turn their attention to the departments of education. The writer, in connection with the Board of Recommendations of the University of Chicago, had occasion recently to ask information from about two hundred institutions with

respect to the training and appointment of teachers. The replies indicated that about fifty colleges and universities either already have, or are about to establish, schools or colleges or departments of education, or are taking steps to strengthen this work where it already exists. It would seem, therefore, with normal schools and certain schools of education devoting themselves to the training of teachers for the elementary schools, and with the colleges and universities awakening to their responsibility for the better preparation of teachers for the secondary schools, that education below the college grade is in a fair way to receive due attention, and with respect to mathematics that it will come in for its share of attention both as to subject-matter and as to the preparation of teachers.

But what of the teaching of college mathematics and the preparation of teachers for work in colleges? Where are the associations devoted to the improvement of teaching mathematics to Freshmen and Sophomores? What universities are offering courses in education especially intended for the preparation of teachers for colleges? Is it to be inferred that special training is regarded as necessary for a secondary teacher of mathematics, but that any one who knows the subject can teach a Freshman? An instructive address* bearing upon this subject was delivered before the Association of Doctors of Philosophy of the University of Chicago last June by Professor Charles Hubbard Judd, the new director of the School of Education at Chicago. The Association had been considering for two years the Relation of the Doctorate to the Teaching Profession, and a symposium† including the opinions of many of the Doctors, together with many important addresses on the subject, have been printed and distributed. Professor Judd's topic was: "The Department of Education in American Universities," and the following scattered sentences will indicate its bearing:

I have been invited to discuss further the subject which you had under consideration last year, namely, the problem of turning Doctors of Philosophy into efficient teachers. There are some of you, I see by the printed reports of your earlier meeting, who regard the ability to teach as a natural gift. There are apparently many of you who are persuaded that courses in pedagogy cannot contribute materially to the improvement of a graduate of a university. In the face of such settled views, backed up by the success that many of you have obtained in the teaching profession, it seems to be a bold and from some points of view a useless undertaking to come before you as I must with the assertion that ability to teach is not a natural gift and that every Doctor of Philosophy would be improved by a careful consideration in a scientific and historical way of the problems of education. * * * * * * *

Until very recently university and college organizations have been based on the opinion expressed by some of you that teaching cannot be made a subject of special study and instruction. Gradually, however, a change is being worked out before our eyes. In spite of opposition and indifference, courses in education are being organized even in the most conservative institutions. * * * * *

The first fact which I wish to point out is that there is a great deal of poor teaching

*Printed in the November, 1909, *School Review*. A few reprints may still be had.

†This has been reprinted and a few copies may still be had by applying to the Secretary of the Association at the University of Chicago.

within our universities and colleges, and this fact can be traced to a neglect upon the part of academic men and women of the form in which they arrange their material. *

It is widely recognized that such a neglect of form as I have indicated appears in much of our university lecturing. The assumption of the ordinary university lecturer is that if he presents a certain body of material so organized that it seems to him to be fairly coherent and logical in its character, it is a matter of small moment whether it appeals to the students because of its literary form or whether it is easily intelligible to them because of its careful adjustment to their present stage of developement. * *

The elective system has in part overcome this attitude and there is a very much more general conviction on the part of college instructors now than there was a generation ago that the demands of the student that the material shall be presented in clear and coherent form should be met. An instructor who is in competition with the other members of his own department for students in his course is likely to recognize the importance of preparing his material as clearly as possible for presentation to his class. * *

It is obvious that everyone who is successful in the art of teaching must have complied with the demands indicated in the foregoing discussion; that is, he must have organized his material in such a way that it has significance not only for his own mind but also for the minds of others. The teacher who does not sympathize with his pupils fails commonly because he does not recognize the type of fact which we have just been discussing.

It should not be inferred from the opening sentence above quoted that all of the five hundred Doctors of the University of Chicago look with distrust upon the proposition that a university man who expects to teach in a college or university should know something of the principles, history and practice of education in general and much of these things as they pertain to his own department in particular, in order that he may, at least, avoid the historic blunders of his predecessors. The following quotation from the symposium bears upon this point:

As matters now stand, whatever the *ideal* held by the university world in regard to the high calling of the doctor in the field of pure research, the *fact* is that the great majority of the doctors now turned out in this country are nominated and pushed by their respective departments, for teaching positions, which they must fill successfully or be counted as failures by all who measure the ratio of results accomplished to tasks undertaken.

Success in teaching is well-nigh *indispensable* (save perhaps in a great university, where failure is less conspicuous because of the large numbers and multifarious interests). If the doctor is to gain encouragement and opportunity to go on with his research, he must first establish a reputation for himself in the community where he goes, for soundness of judgment, clearness of presentation, and power to inspire and lead students; then, having made sure of his ground, he can have pretty much his own way in planning and carrying out his work in the interest of his research. But, on the contrary, if in his blind devotion to his ideal he fails at the outset to teach successfully, to inspire and to lead students, he thereby cuts himself off from his very best resources for ultimately accomplishing his highest aims.

These being the facts as realized in everyday experiences, the conclusion is forced upon us that the universities must prepare for teaching those doctors whom it is proposed to recommend as teachers, and preparations must include breadth of culture, eliminations of angularities, development of pedagogical sense, and some acquaintance with the great educational movements of the past and present. If it be urged that these, some or all of them, are either matters of personal quality or are foreign to the great purpose of the graduate school as director of explorations in unknown fields, then the other conclusion is forced upon us that only those doctors should be recommended as teachers who possess by

nature the prerequisities or have developed them *outside* or *in spite of* the graduate school, and that the present indiscriminate practice of recommending any doctor as a teacher, however narrow or however lacking in the elements of pedagogic sense and power to teach, be superseded by a careful discrimination and selection of those who are prepared to teach and a refusal to nominate or recommend those who are not.

Such a discrimination will lead, as it should, to a more careful and personal consideration of every individual's candidacy for the doctorate, and will compel a readjustment of curriculum and a recognition as a distinct field of that broad, thorough, and scientific training, commensurate with any standard now set up for the doctorate, which is demanded by the man or woman as a preparation for the highest attainment in teaching, leaving undisturbed the narrowest and possibly the deepest channels of pure research to be followed by those who either have not the time or have not the inclination, or perchance have not the personal qualities needed in preparation for teaching.

This leads again and finally to the conclusion either that the number of those who are encouraged to go on to the doctorate should be greatly diminished, or else that the basis of the doctorate should be greatly broadened so as to provide the highest standards and the strongest preparation for the noble art of teaching, as well as to produce fine investigators.

Whatever may have been the attitude of the universities up to this time in regard to training men as teachers for college positions, it is clear that from this time forward the demand is likely to increase for men thus trained, not only because the colleges themselves are awakening to the necessity of better teaching, but also because the irresistible wave of progress in this respect in the secondary schools is bound to reach up into the colleges and shame them into action. Fortunately, however, the agitation is already begun to some extent with respect to the colleges, under the challenge of such men as Abraham Flexner and with the support of many who are loyal to the cause of better teaching.

But while the question of better training for teachers of mathematics is sure to get consideration along with the like questions for all other departments in the general upward movement, it is the special responsibility of the mathematicians themselves to consider the subject-matter of the collegiate curriculum, with respect to its better arrangement, better form of presentation, closer contact with concrete applications, closer correlation with related departments, such as Physics, Astronomy, Chemistry, Geology, and better adjustment to both the earlier and later work in the department. In this connection also belongs the consideration of all these questions concerning mathematics as related to students in the literary courses, in the general science courses, in the specialized science courses, in the small college and in the great university.

It is well known that changes have been going on in respect to the status of various branches of mathematics in the college curriculum; for instance, as to the content of College Algebra, the scope of Trigonometry and of Analytic Geometry, the proper adjustment of theory and applications in the Calculus, and the whole question as to whether all of these branches should not be considered as one subject—Mathematics—rather than as sepa-

rate topics carefully partitioned off from each other both theoretically and pedagogically.

The editors of the MONTHLY have decided to open its columns to contributions on the teaching of collegiate mathematics, in the hope that some impetus may be given to the better training of teachers, the better arrangement and coordination of material and the better form and methods of presentation. Already some important papers have been promised and others will be announced in the near future.

Moreover, no less than six sub-committees, under Klein's International Commission on the Teaching of Mathematics, are now working in this country on topics connected with mathematics of a collegiate grade, and it is supposed that the substance of their reports will be made known in America and will become the basis of discussion at the various meetings of mathematical societies during the Autumn and Winter. From these sources also much should be expected both in pointing out existing conditions and in arousing interest in questions of possible improvements.

ON A FEW POINTS IN THE HISTORY OF ELEMENTARY MATHEMATICS.

By G. A. MILLER, University of Illinois.

The main object of the present note is to call attention to the recent changes of view in reference to several important questions in the history of elementary mathematics. Among the general works on the history of mathematics there is probably none which is more commonly trusted than Cantor's *Vorlesungen über Geschichte der Mathematik.* The first volume of this work is now in its third edition, the second and third volumes are in second editions, while the first edition of the fourth volume appeared as recently as 1908. As the first edition of the first volume appeared in 1880, the rapid succession of new editions of such an extensive work is an index of the rapid recent progress in the development of the history of mathematics. Discoveries have followed each other in such close succession as to call for rapid changes of view even in regard to some of the most fundamental matters.

In support of this statement we shall give a few changes of view as exhibited in the second and third editions of volume I of Cantor's monumental work. On page 576 of the second edition the following words may be found, "According to our opinion the discovery of zero is due to the Hindus." The corresponding statement in the third edition, page 616, reads as follows: "According to our opinion the discovery of zero is due to the Baby-

lonians, the deepening of the concept is due to the Hindus." It need scarcely be added that the former of these two views is expressed in nearly all works of reference relating to this subject, but it is likely that this will gradually be changed on account of the great influence of Cantor in historical matters and the strong reasons advanced by him in support of his change of view.

The discovery of zero, as used above, implies its use in positional arithmetic. It is certain that the Greeks employed zero in the second century B. C. to denote the absence of degrees, minutes, or seconds in their sexagesimal notation.* The earliest known use of this symbol in Babylonian inscriptions belongs to the third century B. C., but it is supposed that it was in use at a much earlier date. At the international mathematical congress held in Paris in 1900 Cantor suggested that zero was probably in use among the Babylonians as early as 1700 B. C. Even if such an early date cannot be established it appears likely that scholars will hereafter attribute the discovery of positional arithmetic to the Babylonians instead of to the Hindus.

Another important change of view exhibited in the two editions mentioned above relates to the sexagesimal system of notation, which is still used by us in measuring time and angles. In the second edition, page 92, Cantor says that the ancient Babylonian astronomers probably thought the year was composed of 360 days and hence they divided the circle into 360 parts, each part corresponding to a day of the year. An apparent corroboration of this view was furnished by an ancient Chinese custom to divide the circle into 365¼ degrees. After the circle was divided into 360°, it is easy to see that the base 60, for the sexagesimal system, might have been suggested by the fact that the side of a regular inscribed hexagon is equal to the radius of the circle.

In the third edition Cantor abandons this view and suggests that the base 60 may have resulted from the union of two nations, meeting in Babylon, one having a system of numeration with 10 as a base while the other employed 60 as a base. Among the reasons advanced for abandoning his former position are the following: The counting and writing of small numbers must have been known long before such comparatively large numbers as 360 were in use, and it must have been recognized by the early Babylonian astronomers that the year involves more than 360 days. Hence it does not appear likely that there is any connection between the number of degrees in a circle and the supposed number of days in the year.

These changes of view should influence the high school teacher to supplement the historical notes even in many of the best and most recent text-books. Whenever the teacher expresses an opinion in reference to a question which has not been fully settled this opinion should be in accord with the most advanced scholarship. The changes of view of such an eminent scholar as Moritz Cantor should also impress the teacher with the fact

*_Encyclopedie des Sciences Mathematiques_, Vol. 1, page 17.

that the most modern works should be consulted when one is dealing with historical data. What the foremost scholar along a certain line endorsed ten years ago may be antiquated rubbish to-day.

A less positive but scarcely less important change of view, exhibited in the two editions under consideration, relates to the development of Egyptian mathematics at the time of Ahmes, about 1700 B. C. As the extraction of the square root does not occur in the work of Ahmes, it was natural to infer that the Egyptians were unacquainted with this operation at this early date. In the third edition we are told, on the contrary, that the ancient Egyptians knew how to extract the square root and even how to solve such systems of simultaneous equations as

$$x^2+y^2=100$$
$$x:y=1:\tfrac{3}{4}.$$

These recent discoveries prove that the ancient Egyptians knew much more about equations than what was previously supposed.

Recent discoveries in Babylon have also led to changes of view. Many of the histories of mathematics state that the Babylonians did not use any number as large as a million and some historians have devoted considerable attention to this supposed fact. The recent discoveries by Hilprecht of the University of Pennsylvania have revealed that the Babylonians made use of as large numbers as

195, 955, 500, 000, 000

and that numbers exceeding a million were frequently employed.* Even in the latest edition of Cantor's *Vorlesungen* the obsolete view is expressed.

In closing this note it may be in place to quote the remarks of J. J. Milne before a meeting of British teachers of mathematics, as follows: "Speaking as a schoolmaster to schoolmasters I think we ought to bring the history of mathematics more than we do before the notice of our pupils. Mathematics is a living, growing science, with a definite history, and there is not a branch of it which boys take up in school, whether arithmetic or algebra, or geometry, or trigonometry, or any other of the many divisions, but has its own history, and I have always found that boys are interested in learning what properties were known to the ancients, and what have been discovered in modern times, and I often think that the writers of text-books would do well to devote a little more space than they do to what I may call the note of human interest." For instance, in teaching complex fractions it is of interest to observe that the Hindus of the ninth century taught the rule of inverted divisor and that this rule was rediscovered in Europe in the sixteenth century.

*Smith, *Bulletin of the American Mathematical Society*, vol. 13, page 394.

ON CERTAIN FUNCTIONAL EQUATIONS.*

By R. D. CARMICHAEL, Princeton University.

1. Let $h(x)$ be a uniform analytic function regular in the neighborhood of the origin and defined by the equation

$$h(x+y).h(x-y)=h(x)^2+h(y)^2-c^2,\ c\neq 0. \tag{1}$$

If we put $x=y=0$, we have

$$h(0)^2=2h(0)^2-c^2, \text{ or } \pm h(0)=c. \tag{2}$$

If now we put $\pm h(x)=cf(x)$, the preceding relations become

$$f(x+y).f(x-y)=f(x)^2+f(y)^2-1, \tag{3}$$
$$f(0)=1. \tag{4}$$

Hence, putting $x=y$,

$$f(2x)=2f(x)^2-1. \tag{5}$$

Then

$$f'(2x)=2f(x)f'(x). \tag{6}$$

Differentiating (3) with respect to x and then with respect to y,

$$f'(x+y).f(x-y)+f(x+y).f'(x-y)=2f(x)f'(x), =f'(2x), \text{ by (6)}; \tag{7}$$
$$f'(x+y).f(x-y)-f(x+y).f'(x-y)=2f(y)f'(y), =f'(2y), \text{ by (6)}. \tag{8}$$

If we put $x+y=A$, $x-y=B$, whence $2x=A+B$, $2y=A-B$, and substitute in equations (7) and (8), and finally in the result replace A and B by x and y, respectively, we have the equations

$$f'(x+y)=f'(x)f(y)+f(x)f'(y); \tag{9}$$
$$f'(x-y)=f'(x)f(y)-f(x)f'(y). \tag{10}$$

Adding (9) and (10) and integrating the result with respect to x we obtain

$$f(x+y)+f(x-y)=2f(x)f(y); \tag{11}$$

for from equation (4) it follows that the constant of integration is zero.

We will now show how to revert from equation (11) to equations (1) and (3), when $f(0)\neq 0$. Differentiate (11) with respect to x and then with

*Presented to the American Mathematical Society, April 24, 1909.

respect to y. From the addition and the subtraction of the two resulting equations follow (9) and (10), respectively. In (9) put $x=y$; the result is equation (6). Then, if in (9) and (10) we put $x+y$ for x and $x-y$ for y and employ (6), we obtain (7) and (8). If we integrate (7) and (8) with respect to x and y, respectively, combine the results and choose $-a$ as a constant of integration, we have $f(x+y).f(x-y)=f(x)^2+f(y)^2-a$, where $a\neq 0$ since $f(0)\neq 0$.

From the foregoing considerations it follows that, except possibly for the appearance of a constant factor, equations (1), (3), (11) have the same solution (or solutions) when each alone is taken as a single functional equation, the condition $f(0)\neq 0$ being satisfied. Moreover, either the system (9) and (10) or the system (7) and (8) has the same solution if $f(0)\neq 0$. Then if general f is found subject to the above equations and multiplied by a proper constant, the result is the general solution of (1), (3), (11), (7) and (8), or (9) and (10).

We proceed now to find general f in equation (3). All the equations (4) to (11) depend on (3) and the fact that $f(x)$ is differentiable; and we may therefore make use of any of them.

We first show that $f(x)$ is an even function. Putting $y=-x$ in (3) we have

$$f(2x)=f(x)^2+f(-x)^2-1.$$

Comparing with (5),

$$f(x)^2=f(-x)^2;\ \text{or}\ f(x)=f(-x),\ \text{since}\ f(0)\neq 0.$$

Since $f(0)=1$ and $f(x)$ is an even function regular in the neighborhood of the origin, it may be expanded in series in the form

$$f(x)=1+\frac{ax^2}{2!}+\frac{bx^4}{4!}+\frac{cx^6}{6!}+\frac{dx^8}{8!}+\ldots.$$

Substituting in (5) to determine coefficients, we have $b=a^2$, $c=a^3$, $d=a^4$, ... Hence, writing k^2 for a, we have

$$f(x)=\tfrac{1}{2}(e^{kx}+e^{-kx}), \tag{12}$$

as the general solution of (3) subject to the imposed conditions. Then

$$h(x)=\pm\frac{c}{2}(e^{kx}+e^{-kx}) \tag{13}$$

is the solution of (1) under like conditions. This completes the discussion for the case $c\neq 0$.

If $c=0$, equations (1) and (2) become

$$h(x+y).h(x-y)=h(x)^2+h(y)^2, \qquad (14)$$
$$h(0)=0.$$

Differentiating with respect to x and then with respect to y, we have

$$h'(x+y).h(x-y)+h(x+y).h'(x-y)=2h(x)h'(x),$$
$$h'(x+y).h(x-y)-h(x+y).h'(x-y)=2h(y)h'(y).$$

Adding, putting $y=x$, and remembering that $h(0)=0$, we have

$$h(x).h'(x)=0.$$

Hence $h(x)=0$ or $h'(x)=0$. In either case $h(x)$ is a constant. Since $h(0)=0$, that constant is zero. Hence $h(x)=0$, and the case is trivial. Therefore the function $\frac{1}{2}(e^{kx}+e^{-kx})$, multiplied by a proper constant, constitutes the significant solution of our equations. The value of $f(0)$ determines this multiplicative constant.

2. Consider the equation

$$g(x+y).g(x-y)=g(x)^2-g(y)^2. \qquad (15)$$

If we put $x=y=0$, we have $g(0)=0$. Differentiating with respect to x and then with respect to y,

$$g'(x+y).g(x-y)+g(x+y).g'(x-y)=2g(x)g'(x), \qquad (16)$$
$$g'(x+y).g(x-y)-g(x+y).g'(x-y)=-2g(y)g'(y). \qquad (17)$$

Making $x=y$ and remembering that $g(0)=0$, we have

$$g(2x)g'(0)=2g(x)g'(x).$$

In view of this relation the product of (16) and (17) gives

$$g'(x+y)^2.g(x-y)^2-g(x+y)^2g'(x-y)^2=-g'(0)^2g(2x)g(2y);$$

or, $$g(x)^2g'(y)^2-g'(x)^2g(y)^2=g'(0)^2.g(x+y).g(x-y),$$

on replacing $x+y$ by x, $x-y$ by y, $2x$ by $x+y$, $2y$ by $x-y$. Hence in view of (15),

$$g(x)^2g'(y)^2-g'(x)^2g(y)^2=g'(0)^2[g(x)^2-g(y)^2]. \qquad (18)$$

If two equations are formed, respectively, by adding and by subtracting equations (16) and (17) and their product member by member is taken, the result readily reduces to

$$g(x+y).g(x-y).g'(x+y).g'(x-y)=g(x)^2g'(x)^2-g(y)^2g'(y)^2.$$

Divide the members of this equation by those of (15); then

$$g'(x+y).g'(x-y)=\frac{g(x)^2g'(x)^2-g(y)^2g'(y)^2}{g(x)^2-g(y)^2}$$

$$=g'(x)^2+g'(y)^2-\frac{g(x)^2g'(y)^2-g(y)^2g'(x)^2}{g(x)^2-g(y)^2}$$

$$=g'(x)^2+g'(y)^2-g'(0)^2,$$

in view of (18). Comparing this with (1) and (13) we have

$$g'(x)=\pm\frac{g'(0)}{2}(e^{kx}+e^{-kx}).$$

Therefore,

$$g(x)=\pm\frac{g'(0)}{2k}(e^{kx}-e^{-kx}, \text{ if } k=0, \tag{19}$$

since by (15) it is apparent that the constant of integration is zero. But

$$g(x)=\pm g'(0).x, \text{ if } k=0.$$

Therefore (15) has the two solutions (19) and (20), where $g'(0)$ is an arbitrary constant.

ALABAMA PRESBYTERIAN COLLEGE, *April, 1909.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

316. Proposed by B. F. FINKEL, Ph. D.

To prove that $\sum_1^n (-1)^{r-1}\frac{{}_rC_n}{r}=\sum_1^n\frac{1}{r}$.

II. Solution by S. LEFSEHETZ, East Pittsburg, Pa.

The proposition being true for $n=1$ and 2, the following is a proof by induction.

Suppose that $\sum_1^{n-1} (-1)^{r-1}\frac{1}{r}\cdot{}_rC_{n-1}=\sum_1^{n-1}\frac{1}{r}\ldots(1)$.

Since ${}_rC_{n-1}=\frac{n-r}{n}{}_rC_n$, therefore $\frac{1}{r}{}_rC_{r-1}=\frac{1}{r}{}_rC_n-\frac{1}{n}{}_1C_n$.

$\therefore\sum_1^{n-1}(-1)^{r-1}\frac{1}{r}{}_rC_{n-1}=\sum_1^{n-1}(-1)^{r-1}\frac{1}{r}{}_rC_n-\frac{1}{n}\sum_1^{n-1}(-1)^{r-1}\,{}_rC_n=\sum_1^{n-1}\frac{1}{r}\ldots(2)$.

But $(1-1)^n=0=-\sum_1^{n-1}(-1)^{r-1}\,{}_rC_n-(-1)^{n-1}\,{}_nC_n+1$.

$\therefore-\sum_1^{n-1}(-1)^{r-1}\,{}_rC_n=(-1)^{n-1}\,{}_nC_n-1$.

By substituting in (2), we have, $\sum_1^n(-1)^{r-1}\frac{1}{r}{}_rC_n=\sum_1^n\frac{1}{r}$.

321. Proposed by C. C. BLAND, Attorney at Law, Rolla, Mo.

A corporation is capitalized for \$20,000. 125 shares of the par value of \$100 per share has been issued. A has 27 19/78 shares. B, C, D, E, and F each have 19 43/78 shares. It is the wish of the corporation to cancel the certificates held by A, B, C, D, E, and F, and to issue new certificates to each of them in lieu of those now held by them, and to avoid the issuance of any certificate for a fraction of a share. How many shares should each receive, the whole not to exceed 200, at the same time maintaining the present interest of each in the corporation?.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The value of $27\frac{19}{78}$ shares$=\$2724\frac{14}{39}=\$\frac{106250}{39}$, and the value of $19\frac{43}{78}$ shares$=\$1955\frac{5}{39}=\$\frac{76250}{39}$.

The greatest common divisor of these values is $\frac{1250}{39}$.

$\therefore \$\frac{1250}{39}=\$32\frac{2}{39}$ is the highest par value of each share in order that each may have whole shares and maintain his present interest.

A will then have 85 shares, B, C, D, E and E each 61 shares. Total number of shares in corporation, 624.

322. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Take six consecutive prime numbers, as 53, 59, 61, 67, 71, and 73, and find the least whole number such that if it be divided by 59 the remainder will be 53, if it be divided by 67 the remainder will be 61, and if it be divided by 73 the remainder will be 71, and show that this least whole number and the succeeding consecutive whole numbers that will fulfill this condition as to divisions and remainders are in arithmetical progression; and also show whether or not this is a general law for n consecutive prime numbers; and if there be such a general law whether or not that general law will lead to a general law for the finding of prime numbers.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let x=the number. Then $\frac{x-53}{59}$, $\frac{x-61}{67}$, $\frac{x-71}{73}$=whole number.

Proceeding in the usual way, we get $x=27665-288569n$.

Therefore 27665 is the smallest whole number satisfying the condition. The succeeding numbers we get by putting $n=-1, -2, -3, \ldots$ which, of course, form an arithmetical progression the common difference of which is 289569. This will always be the case for any number of consecutive prime numbers, though it will not lead to a general law for the finding of prime numbers.

GEOMETRY.

347. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

ABC is a triangle, and D, E, F are the mid points of the arcs of its nine-point circle cut off by BC, CA, AB, respectively. The inscribed circle touches these sides at X, Y, Z. Are the lines DX, EY, FZ concurrent? A purely geometrical discussion required.

No solution of this problem has been received.

348. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Two parabolas and a rectangular hyperbola circumscribe a given quadrilateral. Find a relation between the squares of the latera recta of the parabolas and the squares of the perpendiculars from the center of the hyperbola to the axes of the parabolas.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

(1)...$m(x^2+y^2)+2nxy+2px+2qy+r=0$ is the equation of the rectangular hyperbola;

(2)...$(ax+by)^2+2gx+2fy+c=0$ is the equation of one parabola, and

(3)...$(Ax+By)^2+2Gx+2Fy+C=0$ is the equation of the other parabola

(4)... $[2(fa-gb)\sin^2\beta]/[a^2+b^2-2ab\cos\beta]^{\frac{3}{2}}$ is the latus rectum for (2);
(5)... $[2(FA-GB)\sin^2\rho]/[A^2+B^2-2AB\cos\rho]^{\frac{3}{2}}$ is the latus rectum for (3).
(6)... $(nq-mp)/(n^2-m^2)$, $(np-mq)/(n^2-m^2)$ is the center of (1).
(7)... $(a+b)(ax+by)+ag+bf=0$ is the axis of (2), and
(8)... $(A+B)(Ax+By)+AG+BF=0$ is the axis of (3).

β, ρ are the angles of inclination of the axes of (2), (3), respectively.

Let $PQRS$ be the quadrilateral; PQ, SR intersecting in the point T at an angle ω. Take T for the origin of all curves, and let $TP=h$, $TQ=h_1$, $TS=k$, $TR=k_1$. Then $\beta=\rho=\omega$.

(9) For (2), $h+h_1=-2g/a^2$, $hh_1=c/a^2$, $k+k_1=-2f/b^2$, $kk_1=c/b^2$.

(10) For (3), $h+h_1=-2G/A^2$, $hh_1=C/A^2$, $k+k_1=-2F/B^2$, $kk_1=C/B^2$.

(11) For (1), $h+h_1=-2p/m$, $hh_1=r/m$, $k+k_1=-2q/m$, $kk_1=r/m$.

$\therefore hh_1=kk_1$, $a=b$, $A=B$.

$g=-a^2(h+h_1)/2$, $f=-a^2(k+k_1)/2$, $G=-A^2(h+h_1)/2$, $F=-A^2(k+k_1)/2$. These values of g, f; G, F in (4), (5), respectively, give the same latus rectum for each. It is

$$\frac{(k+k_1-h-h_1)\sin^2\omega}{[2(1-\cos\omega)]^{\frac{3}{2}}}=\tfrac{1}{2}(k+k_1-h-h_1)\cot\tfrac{1}{2}\omega\,\operatorname{cosec}\tfrac{1}{2}\omega$$

$$=[(k-h)(h-k_1)\cot\tfrac{1}{2}\omega\,\operatorname{cosec}\tfrac{1}{2}\omega]/2h.$$

The square of the perpendicular from (6) on (7) is

$$(12)\ldots\frac{\left[\dfrac{p+q}{n+m}+\dfrac{g+f}{2a}\right]^2\sin^2\omega}{2(1-\cos\omega)}=\tfrac{1}{4}(h+h_1+k+k_1)^2\left(\frac{m}{n+m}+\frac{a}{2}\right)^2\cos^2\tfrac{1}{2}\omega.$$

The square of the perpendicular from (6) on (8) is

$$(13)\ldots\frac{\left[\dfrac{p+q}{n+m}+\dfrac{G+F}{2A}\right]^2\sin^2\omega}{2(1-\cos\omega)}=\tfrac{1}{4}(h+h_1+k+k_1)^2\left(\frac{m}{n+m}+\frac{A}{2}\right)^2\cos^2\tfrac{1}{2}\omega.$$

(12) $\div$ (13) gives $\left[\dfrac{m(2+a)+an}{m(2+A)+An}\right]^2$ for the ratio.

CALCULUS.

279. Proposed by L. H. McDONALD, M. A., Ph. D., Sometime Tutor at Cambridge, Jersey City, N. J.

Find the ellipse of minimum area which will pass through the vertices of a triangle. (Hedrick-Goursat's *Math. Anal.*, p. 133, ex. 9.)

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

If B be the origin; BA, BC the axes; then from Vol. X, Nos. 8-9, of the MONTHLY, page 204,

$$clx^2+any^2+(al+cn-bm)xy-aclx-acny=0,$$

is the equation to the circum-ellipse, and

$$\frac{2\pi a^2bc^2lmn\sin B}{[4acln-(al+cn-bm)^2]^{\frac{3}{2}}}=\text{its area.}$$

Let $m=ul$, $n=vl$. $\therefore cx^2+avy^2+(a+cv-bu)xy-acx-acvy=0 \ldots (1)$.

$$\frac{2\pi a^2bc^2uv\sin B}{[4acv-(a+cv-bu)^2]^{\frac{3}{2}}}=m=\text{minimum.}$$

Now $dm/du=0$ and $dm/dv=0$.

$$\therefore 4acv-(a+cv-bu)^2=3bvu(a+cv-bu), \ldots (2).$$
$$4acv-(a+cv-bu)^2=3cvu(a-cv+bu). \ldots (3).$$

Hence, $b(b+c)u=c(b+c)v+a(b-c)$, or $u=\dfrac{cv}{b}+\dfrac{a(b-c)}{b(b+c)} \ldots (4)$.

(4) in (2) gives, $3cv^2(b+c)-v[2(b+c)^2-3a(b-c)]+2ac=0$.

$$\therefore v=\frac{2(b+c)^2-3a(b-c)-\sqrt{[2(b+c)^2-3a(b-c)]^2-24ac^2(b+c)}}{6c(b+c)}.$$

$$u=\frac{2(b+c)^2+3a(b-c)-\sqrt{[2(b+c)^2-3a(b-c)]^2-24ac^2(b+c)}}{6c(b+c)}.$$

These values of u and v in (1) give the required ellipse.

Corollary. If $a=b=c$, $u=v=\frac{1}{3}$.

II. **Solution by J. SCHEFFER, A. M.. Hagerstown, Md.**

Let ACB be the triangle; choose $BC=a$ for the axis of abscisses, and CA for that of ordinates. Any circumscribed ellipse is of the form $y^2+Axy+Bx^2+Cy+Dx=0$; and since $(a, 0)$ and $(0, b)$ are points of the ellipse, we have $C=-b$, $D=-Ba$, and the above equation reduces to $y^2+Axy+Bx^2-by-Bax=0$.

Transforming it to the center of the ellipse as origin, it reduces to

$$y^2+Axy+Bx^2-\frac{Bb^2-AB\,ab+B^2a}{4B-A^2}=0.$$

The area is $=\pi\sin C.\ \dfrac{Bb^2-AB\,ab+B^2a}{4B-A^2}=m$. Developing $\dfrac{\partial m}{\partial A}=0$, and $\dfrac{\partial m}{\partial B}=0$, we find $A=\dfrac{b}{a}$ and $B=\dfrac{b^2}{a^2}$, and thus find the ellipse of minimum area to be, $a^2y^2+abxy+b^2x^2-a^2by-ab^2x=0$. The center is the point $(a/3, b/3)$. The maximum ellipse about the triangle is concentric, and its equation is

$$a^2y^2+abxy+b^2x^2-a^2by-ab^2x+\frac{a^2b^2}{4}=0.$$

Also solved by C. N. Schmall, and V. M. Spunar.

280. **Proposed by C. N. SCHMALL, 89 Columbia Street, New York.**

Find the envelope of the system of spheres

$$\left.\begin{aligned}(x-a)^2+(y-b)^2+z^2=r^2\\ a^2+b^2=c^2\end{aligned}\right\}.$$

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; V. M. SPUNAR, Pittsburg, Pa., and the PROPOSER.

Differentiating $(x-a)^2+(y-b)^2+z^2=r^2$ and $a^2+b^2=c^2$ with reference to a as the independent variable, we have $(x-a)+(y-b)\dfrac{\partial b}{\partial a}=0$, and $a+b\,\dfrac{\partial b}{\partial a}=0$; from the second equation we get $\dfrac{\partial b}{\partial a}=-\dfrac{a}{b}$, and substituting in the first we get $(x-a)-\dfrac{a}{b}(y-b)=0$, whence $b=\dfrac{ay}{x}$ and combining this with $a^2+b^2=c^2$, we get $a^2=\dfrac{c^2x^2}{x^2+y^2}$, $b^2=\dfrac{c^2y^2}{x^2+y^2}$, and substituting this in the first given equation, we have, after some easy reductions:

$$x^2+y^2+z^2-2c\sqrt{(x^2+y^2)}=r^2-c^2.$$

This is, as can easily be seen, the equation of the surface of a ring, the central line of which is a circumference whose radius is $=c$, and the perpendicular section of a circle whose radius is $=\sqrt{(2c^2-r^2)}$.

MECHANICS.

229. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the area being vertical, the bounding axis major being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

Solution by the PROPOSER.

Let AB be the intersection of the vertical plane with the surface of the water; AC the major axis of the ellipse$=2a$; Q any point on the area of the semi-ellipse; $QD=h$, the perpendicular distance from Q to AB; $\angle BAC=\beta$, $\angle CAQ=\theta$, $AQ=r$. Then

$$r=\frac{2ab^2\cos\theta}{a^2\sin^2\theta+b^2\cos^2\theta},\quad h=r\sin(\theta+\beta).$$

$$\bar{x}=\frac{\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\cos\theta\, d\theta\, dr}{\int_0^{\frac{1}{2}\pi}\int_0^r hr\, d\theta\, dr},\quad \bar{y}=\frac{\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\sin\theta\, d\theta\, dr}{\int_0^{\frac{1}{2}\pi}\int_0^r hr\, d\theta\, dr}.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^r hr\, d\theta\, dr=\int_0^{\frac{1}{2}\pi}\int_0^r r^2\sin(\theta+\beta)\, d\theta\, dr=\tfrac{1}{3}\int_0^{\frac{1}{2}\pi} r^3\sin(\theta+\beta)\, d\theta=N.$$

$$\therefore N=\frac{8a^3b^6}{3}\int_0^{\frac{1}{2}\pi}\frac{(\cos^3\theta\ \text{tin}\ \theta\cos\beta+\cos^4\theta\sin\beta)\, d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^3}$$

$$=\tfrac{1}{6}ab(4b\cos\beta+3a\pi\sin\beta).$$

$$M=\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\cos\theta\, d\theta\, dr=\int_0^{\frac{1}{2}\pi}\int_0^r r^3\cos\theta\sin(\theta+\beta)\, d\theta\, dr$$

$$=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi} r^4\cos\theta\sin(\theta+\beta)\, d\theta$$

$$=4a^4b^8\int_0^{\frac{1}{2}\pi}\frac{(\cos^5\theta\sin\theta\cos\beta+\cos^6\theta\sin\beta)\, d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^4}$$

$$=\tfrac{1}{24}a^2b(16b\cos\beta+15a\pi\sin\beta).$$

$$P=\int_0^{\frac{1}{2}\pi}\int_0^r hr^2\sin\theta\, d\theta\, dr=\int_0^{\frac{1}{2}\pi}\int_0^r r^3\sin\theta\sin(\theta+\beta)\, d\theta\, dr$$

$$=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi} r^4 \sin\theta \sin(\theta+\beta)\, d\theta$$

$$=4a^4b^8\int_0^{\frac{1}{2}\pi}\frac{(\cos^4\theta\sin^2\theta\cos\beta+\cos^5\theta\sin\theta\sin\beta)\,d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^4},$$

$$=\tfrac{1}{24}ab^2(16a\sin\beta=3\pi b\cos\beta).$$

$$\bar{x}=M/N=\tfrac{1}{4}a.\frac{16b\cos\beta+15a\pi\sin\beta}{4b\cos\beta+3a\pi\sin\beta},\quad \bar{y}=P/N=\tfrac{1}{4}b.\frac{16a\sin\beta+3b\pi\cos\beta}{4b\cos\beta+3a\pi\sin\beta}.$$

If $\beta=0$, $\bar{x}=a$, $\bar{y}=3\pi b/16$.

Also solved by **S. Lefsehetz.**

230. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A particle is projected from a distance $a=2r$ from the earth's center towards the earth with a velocity from infinity. If the earth were an airless homogeneous sphere, radius equal to the present mean radius and gravity as at present, with what velocity and in what time would it reach the center through an opening from surface to center?

Solution by the PROPOSER.

$\int_{v_1}^{v} vdv=\int_x^a fdx$, but $f=\dfrac{gr^2}{x^2}$ above the earth where r is earth's radius.

$$\therefore\int_{v_1}^{v} vdv=gr^2=\int_x^a\frac{dx}{x^2}.\quad \therefore\tfrac{1}{2}(v^2-v_1^2)=\frac{gr^2\ (a-x)}{ax}.$$

$$\therefore v^2=\frac{2gr^2\ (a-x)}{ax}+v_1^2=\left(\frac{dx}{dt}\right)^2,\ \text{where } v_1 \text{ is the initial velocity.}$$

$$\therefore t=\sqrt{a}\int_r^a\frac{\sqrt{(x)}\,dx}{\sqrt{[2agr^2+x(av_1^2-2gr^2)}}.$$

If $v_1=r\sqrt{\dfrac{2g}{a}}$, $t=\dfrac{2(a^{\frac{3}{2}}-r^{\frac{3}{2}})}{3r\sqrt{(2g)}}$, $v=\sqrt{(2gr)}$ at surface.

If $v_1<r\sqrt{\dfrac{2g}{a}}$, $t=2gr^2\sqrt{\left(\dfrac{a}{2gr^2-av_1^2}\right)^3}\left(\cos^{-1}\sqrt{\dfrac{2gr^2-av_1^2}{2agr}}\right.$

$$-\cos^{-1}\sqrt{\frac{2gr^2-av_1^2}{2gr^2}}+\frac{\sqrt{[(2gr^2-av_1^2)(2agr+av_1^2-2gr^2)]}}{2agr}$$

$$-\frac{v_1\sqrt{[a(2gr^2-av_1^2)]}}{2gr^2}\Bigg).$$

If $v_1<r\sqrt{\frac{2g}{a}}$, $t=2gr^2\sqrt{\left(\frac{a}{av_1^2-2gr^2}\right)^3}\Bigg\{\frac{v_1\sqrt{[a(av_1^2-2gr^2)]}}{2gr^2}$

$$-\frac{\sqrt{[(av_1^2-2gr^2)(2agr+av_1^2-2gr^2)]}}{2agr}$$

$$+\log\left[\sqrt{\frac{a}{r}}\left(\frac{\sqrt{(av_1^2-2gr^2)}+v_1\sqrt{a}}{\sqrt{(av_1^2-2gr^2)}+\sqrt{(2agr+av_1^2-2gr^2)}}\right)\right]\Bigg\}.$$

Since $v_1=\sqrt{(2gr)}=$velocity from infinity, and $a=2r$, $v_1>r\sqrt{(2g/a)}$. Hence time of falling to surface is

$$t=2\sqrt{\frac{r}{g}}\left(\frac{2\sqrt{2}-\sqrt{3}}{2}+\log\frac{2+\sqrt{2}}{1+\sqrt{3}}\right).$$

Velocity at the surface becomes

$$v_2^2=2gr^2\left(\frac{2r-r}{2r^2}\right)+2gr=3gr. \quad \therefore v_2=\sqrt{(3gr)}.$$

The formula for below the earth's surface is

$$\int_{v_2}^{v_3} v_3\,dv=\int f\,dx=\frac{g}{r}\int_0^x (r-x)\,dx. \quad \therefore v_3^2-v_2^2=\frac{g}{r}(2rx-x^2).$$

$$\therefore v_3^2=(g/r)(2rx-x^2)+v_2^2=(dx/dt)^2.$$

$$\therefore t_1=\int_0^r \frac{dx}{\sqrt{[v_2^2+(g/r)(2rx-x^2)]}}=\sqrt{\frac{r}{g}}\sin^{-1}\left(\frac{r\sqrt{g}}{\sqrt{(v_2^2r+gr^2)}}\right).$$

Hence time of falling from surface to center is $t_1=\frac{\pi}{6}\sqrt{\frac{r}{g}}$.

Since $v_2=\sqrt{(3gr)}$, $v_3=2\sqrt{(gr)}$, the velocity at the center when $x=r$.

Taking for the mean semi-axes of the earth the values 20926202 feet and 20854895 feet, the radius of a sphere of equal volume is 20902410 feet, and the acceleration g at the surface of this sphere is 32.10614 feet.

$\therefore \sqrt{(r/g)}=806.871.$

$$T=t+t_1=\sqrt{\frac{r}{g}}\left(\tfrac{1}{6}\pi+2\sqrt{2}-\sqrt{3}+2\log\frac{2+\sqrt{2}}{1+\sqrt{3}}\right)=1666.673 \text{ seconds} = 27$$
minutes, 46.673 seconds.

$v_3=44869.668$ feet per second$=8.498$ miles$=8\frac{1}{2}$ miles per second, nearly.

When $v_1=r\sqrt{(2g/a)}$, $v_2=\sqrt{(2gr)}$, and the velocity of arriving at the surface is independent of a, the distance from the center.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

163. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Prove that the equation $y^n=mx+1$ always has at least one positive integer solution (different from $y=1$, $x=0$), whatever integer values m and n may have.

Solution by S. LEFSEHETZ, Pittsburg, Pa.

The following solution is evident:

$$y=(m+1)^n,\quad x=\frac{(m+1)^n-1}{m}.$$

To find all solutions, we remark that it is enough to find all values of y such that $y^n\equiv1 (\text{mod } m)$. Let f be a divisor of $\phi(m)$, a number to which f appertains. Then $a^f\equiv1$ (mod m). If also $a^n\equiv1$ (mod m), we must have $n\equiv0$ (mod f). Hence, f is a divisor of $dv[n,\ \phi(m)]$. Therefore we take the $\phi(m)$ numbers smaller than m and prime to it, we form the exponents to which they appertain and keep them if their exponents divide $dv[n,\ \phi(m)]$. If x be such a value, $y=x+km$ is a solution, the corresponding value of x being $\dfrac{y^n-1}{m}$, which is integral since $y^n\equiv1$ (mod m).

AVERAGE AND PROBABILITY.

200. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

A line $AB=l$ is extended to P making $BP=p$. If a point D is taken at random in BP, what is the mean value of $AD.DP$?

Solution by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Let $x=BD$. Then $AD=l+x$, $DP=p-x$, and $AD.DP=(l+x)(p-x)$.

$$\therefore M=\int_0^p (l+x)(p-x)dx \Big/ \int_0^p dx=\frac{1}{p}\int_0^p[lp+(p-l)x-x^2]dx$$
$$=\tfrac{1}{6}p(3l+p).$$

Also solved by G. B. M. Zerr.

201. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A random straight line is drawn across a circle and another through a given point on the circumference. Find the chance that they intersect within the circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let θ be the angle the line through a given point on the circumference makes with the diameter through the point, AB=length of this chord.

Then for favorable case of intersection the random line must intersect $AB=2a\cos\theta$ where a=radius of circle.

$$\therefore \text{Chance}=\int_0^{\frac{1}{2}\pi} 2a\cos\theta\, d\theta \Big/ \int_0^{\pi} 2a d\theta = \int_0^{\frac{1}{2}\pi}\cos\theta\, d\theta \Big/ \int_0^{\pi} d\theta = \frac{1}{\pi}.$$

Also solved by the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

326. Proposed by R. D. CARMICHAEL, Princeton University.

Is the series, of which the nth term is $\dfrac{1.3.5.7\ldots(2n-1)}{(n+1)!\,2^n.(2n+3)}$ convergent? If so, find its sum.

327. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

The coefficients of the algebraical equation $f(x)=0$ are all integers. Show that if $f(0)$ and $f(1)$ are both odd numbers, the equation can have no integral roots.

328. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

If $x^2+xy+y^2=3a^2$, find the maximum value of $bx+cy$.

GEOMETRY.

354. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Find the condition that triangles which are circumscribed to one of two confocal parabolas may be inscribed in the other.

355. Proposed by JOHN J. QUINN, New Castle, Pa.

If an indefinite line MC cuts the Y-axis at A a fixed point, and the X-axis at B, and at its extremity C another line $PCDP'$ be pivoted cutting the X-axis at D and extending to P', so that $PC=CD=BC$, and $PD=P'D$: (1) Find the locus of P and P' as MC slides through A; (2) Apply to the trisection of an angle; (3) Prove PP' a constant tangent to upper branch; (4) Show condition which gives rise to loop; (5) Show its relation to conchoid; (6) Discuss for other properties.

356. **Proposed by G. I. HOPKINS, Manchester, N. H.**

Required to construct a triangle having given, base, vertical angle, and difference of other two sides.

CALCULUS.

284. **Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.**

Inscribe the triangle of maximum area in a given circle.

285. **Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.**

If R_1 and R_2 are the radii of curvature of an ellipse at the extremities of a pair of conjugate diameters, show that $R_1^{2/3}+R_2^{2/3}=\frac{a^2+b^2}{(ab)^{2/3}}$, where a, b, are the semi-axes.

286. **Proposed by R. D. CARMICHAEL, Princeton University.**

Solve the differential equation

$$\begin{aligned}&[a_0x^3+a_1x^2y+a_2xy^2+(a_0-a_1+a_2)y^3\\&\qquad+a_3x^2+a_4xy+a_5y^2+a_6x+a_7y+a_8]dx\\&+[a_0y^3+a_1xy^2+a_2x^2y+(a_0-a_1+a_2)x^3\\&\qquad+a_3y^2+a_4xy+a_5x^2+a_6y+a_7x+a_8]dy=0.\end{aligned}$$

MECHANICS.

238. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the bounding major-axis being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

239. **Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.**

A uniform bar of length $2a$ is placed in a sloping position, its lower end on the ground (coefficient of friction being μ), its upper end in the air, the bar being supported by a rough fixed peg (coefficient of friction μ'), against which it rests. If h is the height of the peg from the ground, and if θ be the angle the bar makes with the horizon, when on the point of slipping, prove that θ is to be found from the equation

$$\sin\theta\cos\theta\,[(\mu-\mu')\cos\theta+\sin\theta\,(1+\mu\mu')]=\mu\,h/a.$$

240. **Proposed by S. A. COREY, Hiteman, Iowa.**

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

NOTES AND NEWS.

At the ninth annual meeting of the Central Association of Science and Mathematics Teachers, a committee appointed the year preceding made an extended report on the securing of real, applied problems in algebra and geometry suitable for use in secondary schools. A large number of teachers had contributed problems during the year and these had been printed in *School Science and Mathematics.* It was pointed out, however, that in many such problems the technical information necessary made them unsuitable for young students, and even problems which might be understood by boys would, in most cases, be little adapted to girls. For this reason most technical problems drawn from carpentry, masonry, machine shops, etc., are of doubtful practical application in the schoolroom. But it was also pointed out that problems in geometry drawn from architecture, decorative and ornamental design are less technical in character and are equally well adapted to both boys and girls.

This question, of vital importance in secondary work, is also commanding attention in college courses and is materially affecting the character of college texts, especially in the calculus.

Another committee reported on the question of unified mathematics for secondary schools, a subject which is also of interest in respect to college mathematics. S.

BOOKS.

Descriptive Geometry. A Treatise from a Mathematical Standpoint, together with a Collection of Exercises and Practical Applications. By Victor T. Wilson, M. E., Professor of Drawing and Design, Michigan Agricultural College. 8vo, viii+237 pages, 140 figures. Cloth, $1.50 net. New York: John Wiley & Sons.

Descriptive geometry is essentially a mathematical subject. The application of its principles to the making of working drawings, however, and the modifications which are made to suit the contingencies of practice, have had a tendency to obscure this fact, and like other theoretical subjects it has suffered mutilation in the interest of short cuts to immediate practical uses. But does not technical education, after all, consist chiefly in an equipment of sound theory? It has been the author's purpose to refrain from any attempt to hold the student's interest by clothing a few principles with some immediate practical application, but instead, to present a sound theoretical treatment. How well he has succeeded he leaves others to judge.

The principles are herein formulated under theorems, as to plane and solid geometry; Illustrative problems are solved in accordance with these theorems and special constructions discussed. The plan of, at least, one well known text is followed of dividing all problems in two parts; the first of which is a statement of the geometrical principles and

the theoretical solution called an analysis; the second is a description of a graphic solution, accompanied by a drawing. An important feature is added, however, of giving the statement of the geometrical conditions and the solution in the analysis in a general form, instead of being made to refer to a certain kind of problem exclusively. *From the Preface.*

The Slide Rule. An Elementary Treatise. By J. J. Clark, M. E. (Lehigh), Dean of the Faculty, International Correspondence Schools, Manager of the Text-book Department, International Text-book Co. Cloth, 62 pages. Scranton, Pa.: Technical Supply Co.

In this little volume, are set forth in detail the use and applications of the slide rule. As many practical men are using the slide rule, this little volume will enable them to extend their knowledge of its application. F.

College Algebra. By H. L. Rietz, Ph. D. (Cornell), Assistant Professor of Mathematics, University of Illinois, and A. R. Carthorne, Ph. D. (Goettingen), Associate Professor of Mathematics, University of Illinois. 8vo. Cloth, xiii+261 pages. Price, $1.40. New York: Henry Holt & Co.

The book begins with a review of High School Algebra, for the benefit of the student who, having had his high school algebra two years before entering college, requires a hasty review of first principles. However, it is not all review; some matter, for example, determinants and graphs, is introduced in order to give the student, at the outset, an enlarged conception of the subject. Many problems from physics and engineering are introduced, yet in this particular the work is not overdone. The book is one of merit, and will lend itself readily to successful teaching. F.

A Text-book of General Physics for Colleges: Mechanics and Heat. By J. A. Culler, Ph. D., Professor of Physics, Miama University. 8vo. Cloth, ix+311 pages. Price, $1.80. Philadelphia: J. B. Lippincott & Co.

In this book, the principles of Mechanics and Heat are set forth in clear and simple terms. The type is large and the illustrations good. Reference matter and tables are placed in an appendix. A number of short lists of problems are found where needed to illustrate principles, and the answers are given at the end of the lists. A table of sines, cosines, tangents, etc., are also appended. The book will be serviceable to those teachers who prefer texts dealing with specific topics. F.

Dynamical Theory of the Capture of Satellites and of the Division of Nebulae under the Secular Action of the Resisting Medium. By T. J. J. See. Reprinted from the *Astr. Nach.*, Vol. 181.

The author has attempted to show by mathematical reasoning that the solar system had its origin, not by the formation of the planets and satellites detached from a central mass as was assumed by LaPlace, but that these planets and satellites were captured from without and have since had their orbits reduced in size and rounded up under the secular action of a resisting medium. The article is interesting even though one might not agree with all the assumptions necessary to establish the conclusions to which the author's reasoning leads. F.

ERRATA.

Page 174, line 24, for "have been printed" read "had been printed."
Page 176, line 30, for "like questions" read "like question."
Page 177, line 16, for "improvements" read "improvement."

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AVERAGE AND PROBABILITY.

MISCELLANEOUS.

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NOTE ON A PROOF IN CHRYSTAL'S ALGEBRA.

By T. E. McKINNEY, University of South Dakota.

An analytic proof of the "fundamental theorem of algebra" is given in Chrystal's *Text-book of Algebra*, Part I, 5th Ed., Chap. XII. In view of the character and authority of this admirable treatise it may be worth while to call in question one or two points in this proof.

The argument, in outline, is substantially this: It is first established, in effect, that the rational, integral function $f(w)$ of degree n in w vanishes, if at all, within a circle, S, having a radius R and the origin for center. It is then virtually assumed that within this circle S, $|f(w)|$ has a lower limit L. The further assumption is made that $L>0$, and the argument is arranged to show that this latter assumption involves a contradiction. To do this, a point w is taken within S, such that

$$|f(w)|=L+\varepsilon,\ \varepsilon>0.$$

A neighboring point, $w+h$, is then taken for the purpose of showing that

$$|f(w+h)|<L,$$

when the h is properly chosen. Now, h having been suitably determined, the argument proceeds:

" $\left|\dfrac{f(w+h)}{f(w)}\right|$ will lie between two positive proper fractions, so that

$$\left|\frac{f(w+h)}{f(w)}\right|=1-\mu,$$

where μ is a proper fraction; and we have

$$|f(w+h)|=(1-\mu)\,|f(w)|=(1-\mu)(L+\varepsilon)=L+\varepsilon-\mu(L+\varepsilon) \qquad (7)$$

which, since ϵ may be as small as we please, is less than L by a finite amount."

The statement following equation (7) is based apparently on the mutual independence of ε and μ. But ε and μ are not independent, being related through their mutual dependence on w. Both are functions of w, as are also the limits of the interval to which

$$\left|\frac{f(w+h)}{f(w)}\right|$$

i. e., $1-\mu$ is assigned. It is conceivable, therefore, that so long as ε is not zero

$$\varepsilon>\frac{\mu L}{1-\mu},$$

and hence that

$$L+\varepsilon-\mu(L+\varepsilon)>L, \ \varepsilon>0.$$

The argument, therefore, as given in the *Text-book*, would not seem to justify the conclusion that the assumption, $L>0$, involves a contradiction. Hence the proof does not appear to be conclusive.

But, waiving this objection for a moment, it is not shown that the point $w+h$ lies within S. This is necessary. For, $f(w)$ does not vanish on S or without it. Hence, if $|f(w+h)| < L$ when and only when the point $w+h$ is on the circle S or without it, then the conclusion is that $f(w)$ does not vanish within S—the opposite of that desired.

Further, apparently this proof presupposes the continuity of $f(w)$. For, let it be granted that $f(w)$ is not everywhere continuous within or on S. Then there may be a region of discontinuity associated with the points giving rise to the limit value L. If so, it is conceivable, until the contrary is shown, that there exists a constant η, $\eta>0$, such that for any w, $|f(w)| \neq L$, we have $\varepsilon>\eta$. Then "ϵ may not be as small as we please." Hence we cannot affirm that a point $w+h$ exists for which

$$|f(w+h)| < L.$$

Continuity, at least in the neighborhood of the points giving rise to the limit value L, has been assumed. The deduction, therefore, of the continuity of $f(w)$ from this proof, as is made in Cor. 2, seems hardly legitimate.

Two of the objections just raised may be avoided by proceeding somewhat as follows:*

*No attempt is made to give an exact and complete formulation of a proof of "the fundamental theorem of algebra." Nor does the *Text-book* up to the point under consideration furnish a wholly satisfactory basis for the line of reasoning here adopted.

Let

$$f(w)=\sum_{0}^{n} ia_i w^{n-i}, \qquad | a_0 | > 0,$$

a_i a complex number, be a polynomial in w which does not vanish on a circle S or without it, S having a radius R and the origin for center. Assume first that $f(w)$ has in S a lower limit L; and second that $L>0$. Then there is in S an unending sequence of values,† w,

$$(1) \qquad w_1, w_2, \ldots, w_n, \ldots$$

defining a number or point W within or on S, with which can be associated an unending sequence of positive numbers, ε,

$$(2) \qquad \varepsilon_1, \epsilon_2, \ldots, \varepsilon_n, \ldots, \varepsilon_i \overline{\geq} \varepsilon_{i+1}, \lim_{i \doteq \infty} \varepsilon_i = 0,$$

such that

$$(3) \qquad 0 < | f(w_i) | - L < \varepsilon_i, \quad i=1, 2, \ldots, \infty.$$

Now employing in part Chrystal's notation,

$$(4) \qquad f(w+h)=f(w)+\sum_{1}^{n} {}^{j} B_j h^j,$$

where B_j is a polynomial in w of degree $n-j$. $B_n=a_0$, is different from zero, and independent of w. It may be shown, after the manner of Chap. V, § 16, that B_j, if it vanish at all, cannot vanish for more than $n-j$ values w. Therefore, there are at most $n(n-1)/2$ values, w, for which some B_j or every B_j vanishes. Some of these values w for which the B's vanish may coincide with W, say τ of them, $0 \overline{\leq} \tau \overline{\leq} n(n-1)/2$. Since the number of values, or points, w, for which the B's vanish is finite, it is possible to draw from W as a center a circle S' having on or within it only those vanishing points of the B's coincident with the W, τ in all. The circle S' may lie wholly within or partly without S. Henceforth we are concerned only with the region common to the two circles S and S'. The boundary of this common region we designate by S_1.

Let us further assume that for $w=W$,

†We assume here the continuity of $f(w)$ in the neighborhood of the points giving rise to the limit value L. It would be easy and, perhaps, much simpler to prove, once for all, that every rational integral function of w is continuous everywhere in the finite plane; that its absolute value has a lower limit in every bounded region of this plane; that the absolute value of the function actually assumes this limit value for some point within or on the boundary of this region.

$$B_1=B_2=\ldots=B_{m-1}=0, \qquad |B_m|>0,\ 0<m\leqq n.$$

Since B_m does not vanish for $w=W$ it does not vanish within or on S_1. Inasmuch as B_j, $j=1, 2, \ldots, n$, is a polynymial in w and $|f(w)|>L$ in S_1, then $B_j/f(w)$ does not become infinite in S_1. Hence there is a positive num- D such that in S_1,

$$\text{(5)} \qquad \left|\frac{B_j}{f(w)}\right|<D,\ j=1, 2, \ldots, n.$$

In particular, let

$$\text{(6)} \qquad \left|\frac{B_m}{f(w)}\right|<D_m.$$

Since B_m does not vanish within or on S_1 we assume that within this region $\left|\frac{B_m}{f(w)}\right|$ has a lower limit d_m, $d_m>0$.

From some value of j on, say $j\geqq k$, all terms w_j of the sequence (1) are in S_1. Only the values w_j, $j\geqq k$, will be considered in what follows.

Now in equation (4) replacing w and h by w_l and h_l, we have

$$\text{(7)} \qquad f(w_l+h_l)=f(w_l)+\sum_1^n{}^j B_{j,l}\,h_l{}^j, \quad l\geqq k,$$

where the subscript l in $B_{j,l}$ signifies that the B's are now polynomials in w_l. Let us set

$$\frac{B_{j,l}}{f(w_l)}=b_{j,l}\,(\cos \alpha_{j,l}+i\sin \alpha_{j,l}),$$

$$h_l=r(\cos \theta_l+i\sin \theta_l).$$

Then

$$\text{(8)} \qquad \frac{f(w_l+h_l)}{f(w_l)}=1+\sum_1^n{}^j\, b_{j,l}\,r^j\,\theta_{j,l},$$

where

$$\Theta_{j,l}=\cos(j\,\theta_l+\alpha_{j,l})+i\sin(j\,\theta_l+\alpha_{j,l}).$$

We now assign to θ_l values such that

$$\text{(9)} \qquad \Theta_{m,l}=-1, \quad l\geqq k.$$

Equation (8) becomes, consequently,

$$(10) \qquad \frac{f(w_l+h_l)}{f(w_l)}=1-b_{m,\,l}r^m+\sum_1^n{}^j b_{j,\,l}\,r^j\,\Theta'_{j,\,l}, \quad l \gtreqless k,\ j\neq m,$$

where the accent on $\Theta'_{j,\,l}$ signifies that θ_l now has the value assigned by equation (9). Hence

$$(11) \qquad 1-b_{m,\,l}-\left|\sum_1^n{}^j b_{j,\,l}\,r^j\Theta'_{j,\,l}\right| \leqq \left|\frac{f(w_l+h_l)}{f(w_l)}\right| \leqq 1-b_{m,\,l}\,r^m$$

$$+\left|\sum_1^n{}^j b_{j,\,l}\,r^j\,\Theta'_{j,\,l}\right|, \quad l \gtreqless k,\ j\neq m.$$

By equation (6) and the remark following,

$$(12) \qquad d_m \leqq b_{m,\,l}<D_m; \text{ further,}$$

$$(13) \qquad \left|\sum_{m+1}^n{}^j b_{j,\,l}\,r^j\Theta'_{j,\,l}\right| \leqq \sum_{m+1}^n{}^j b_{j,\,l}\,r^j<D\sum_{m+1}^n{}^j r^j.$$

For brevity, let

$$(14) \qquad \phi_l=\left|\sum_1^{m-1}{}^j b_{j,\,l}\,r^j\Theta'_{j,\,l}\right|.$$

We now take r, $0<r<1$, so that

$$r^{m+1}>r^{m+2}>\ldots>r^n.$$

Hence,

$$(15) \qquad (n-m)r^{m+1}>\sum_{m+1}^n{}^j r^j.$$

The inequalities (12) may now be replaced by

$$(16) \qquad 1-D_mr^m-\phi_l-(n-m)Dr^{m+1}<\left|\frac{f(w_l+h_l)}{f(w_l)}\right|$$

$$<1-d_mr^m+\phi_l+(n-m)Dr^{m+1}.$$

We may now choose r_1, $r_1>0$, so that, $r<r_1$,

$$(17) \qquad 1-D_mr^m-(n-m)Dr^{m+1}>2g,\ g>0, \text{ and a constant;}$$

and likewise, r_2, $r_2>0$, so that, $r<r_2$,

$$(18) \qquad 1-d_mr^m+(n-m)Dr^{m+1}<1-2f,\ f>0, \text{ and a constant.}$$

Hence there is a number r_3, $0<r_3<r_1$, $r_3<r_2$, such that, $r \leqq r_3$, the inequalities (17) and (18) are both satisfied.

Let it be assumed that $b_{j,l}$, $j=1, 2, \ldots, (m-1)$, continually approaches zero when w_l approaches W as a limit. Hence we can choose an integer k_1 so that $l>k_1$,

(19) $$| b_{j,l}\, \Theta'_{j,l} | \leqq f'/m,\ j=1, 2, \ldots, m-1,\ 0 \leqq f' <f,\ f' <g,$$

where f' is a constant. Hence

(20) $$\phi_l \leqq f',\ l>k_1.$$

Therefore, by (17), (18), and (20),

(21) $$g < \left| \frac{f\,(w_l+h_l)}{f(w_l)} \right| < 1-f,\ l>k_1,\ r \leqq r_3.$$

Let

$$\left| \frac{f\,(w_l+h_l)}{f(w_l)} \right| = 1-\mu_l.$$

Then

(22) $$g<1-\mu_l<1-f,\ l>k_1,\ r \leqq r_3,$$

whence

(23) $$1-g>\mu_l>f.$$

Therefore, μ_l is a positive proper fraction which can approach neither zero nor one as a limit when l approaches infinity. Hence

(24) $$| f(w_l+h_l) | = (1-\mu_l)\, | f(w_l) | =(1-\mu_l)\,(L+\epsilon)$$
$$=L+\varepsilon-\mu_l(L+\epsilon),\ l \dotplus k_1,\ 0 \leqq r \leqq r_3.$$

Now, there is an integer k_2 such that for $l>k_2$ and for r, $0<\delta \leqq r \leqq r_3$, where $\delta<r_3$ and is a constant,

(25) $$L+\varepsilon-\mu(L+\varepsilon)<L.$$

Let us now choose a number r_4, $r_4<r_3$, so that

(26) $$0<r_4<R- | w_l | ,\ k_2<l<k_3,\ k_3 \text{ an integer},$$

and further condition δ by the inequality, $\delta<r_4$. Then the inequality (25) holds for every point w_l+h_l such that $k_2<l<k_3$ and $\delta<r<r_4$; further, every such point is in S.

Therefore, L, $L>0$, is not the lower limit of $f(w)$ in S. Hence $L=0$, and $f(w)$ has a root in S.

THE PERFECT MAGIC SQUARES FOR $\mp$1909.*

By THEODORE L. DeLAND, Treasury Department, Washington, D. C.

In an article on Magic Squares in *Our Schoolday Visitor, Mathematical Almanac and Annual,* 1871, Judge Scott of the Ohio Supreme Court divides them into, simple, perfect, complex, compound, triple, quadruple and quintuple magic squares. I will add one more — potential magic squares, or those existing in possibility, not in reality — being too great to construct.

There can be magic squares — and some very interesting ones — that involve both positive and negative numbers, that respond to general laws for their development. The magic squares for the years ∓ 1909 all belong to this class.

Let y=the year; then we have by Zerr's formula†,

$$2y=n[2x\pm(n^2-1)];$$

from which

$$x=[2y\mp(n^3-n)]/2n.$$

We know that $1909=23\times 83$, the product of two primes, from which

I. When $y=1909$, and $n=23$, we have for the least value, $x=-181$; and for the greatest value, $x=347$.

II. When $y=1909$, and $n=83$, we have for the least value, $x=-3421$; and for the greatest value, $x=3457$.

III. When $y=-1909$, and $n=23$, we have for the least value, $x=-347$; and for the greatest value, $x=181$.

IV. When $y=-1909$, and $n=83$, we have for the least value, $x=-3467$; and for the greatest value, $x=3421$.

Judge Scott also claims that if n be a prime number, not less than 5, we may in every such case construct a great variety of perfect magic squares, and gives the general rule for their construction; and concludes with the remark, that the number of such magic squares may be expressed by the formula, $n^2(n-1)^2(n-3)(n-4)$.

The number of systems of magic squares for the years, -1909 and $+1909$ is limited to I, II, III, and IV, given above, and to the variations of each.

By the formula we have for

I. 97,293,680 magic squares for 1909;

II. 292,752,739,520 magic squares for 1909;

III. 97,293,680 magic squares for -1909; and

IV. 292,752,739,520 magic squares for -1909; or a

Total of 585,700,066,400 magic squares for the year $+1909$ and -1909.

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*In the January, 1909, issue of the MONTHLY, magic squares for 1909 were requested. This article is in response to that request.

†Vol. XVI, No. 1, page 2, of the MONTHLY.

We will now develop a perfect magic square for -1909, when $n=23$. The year, -1909, may be interpreted as before Christ. In each series from $-x$ to $+x$, zero must occupy one cell in all of these magic squares.

There is no magic square for positive integers alone for 1909.

The following perfect magic square for -1909 begins with -347 and ends with $+181$; but the corresponding magic square for $+1909$ would begin with -181 and end with $+347$.

MAGIC SQUARE FOR THE YEAR, —1909.

-70	-45	-20	5	30	55	80	105	130	155	180	-347	-322	-297	-272	-247	-222	-197	-172	-147	-122	-97	-72
-46	-21	4	29	54	79	104	129	154	179	-325	-323	-298	-273	-248	-223	-198	-173	-148	-123	-98	-73	-71
-22	3	28	53	78	103	128	153	178	-326	-324	-299	-274	-249	-224	-199	-174	-149	-124	-99	-74	-49	-47
2	27	52	77	102	127	152	177	-327	-302	-300	-275	-250	-225	-200	-175	-150	-125	-100	-75	-50	-48	-23
26	51	76	101	126	151	176	-328	-303	-301	-276	-251	-226	-201	-176	-151	-126	-101	-76	-51	-26	-24	1
50	75	100	125	150	175	-329	-304	-279	-277	-252	-227	-202	-177	-152	-127	-102	-77	-52	-27	-25	0	25
74	99	124	149	174	-330	-305	-280	-278	-253	-228	-203	-178	-153	-128	-103	-78	-53	-28	-3	-1	24	49
98	123	148	173	-331	-306	-281	-256	-254	-229	-204	-179	-154	-129	-104	-79	-54	-29	-4	-2	23	48	73
122	147	172	-332	-307	-282	-257	-255	-230	-205	-180	-155	-130	-105	-80	-55	-30	-5	20	22	47	72	97
146	171	-333	-308	-283	-258	-233	-231	-206	-181	-156	-131	-106	-81	-56	-31	-6	19	21	46	71	96	121
170	-334	-309	-284	-259	-234	-232	-207	-182	-157	-132	-107	-82	-57	-32	-7	18	43	45	70	95	120	145
-335	-310	-285	-260	-235	-210	-208	-183	-158	-133	-108	-83	-58	-33	-8	17	42	44	69	94	119	144	169
-311	-286	-261	-236	-211	-209	-184	-159	-134	-109	-84	-59	-34	-9	16	41	66	68	93	118	143	168	-336
-287	-262	-237	-212	-187	-185	-160	-135	-110	-85	-60	-35	-10	15	40	65	67	92	117	142	167	-337	-312
-263	-238	-213	-188	-186	-161	-136	-111	-86	-61	-36	-11	14	39	64	89	91	116	141	166	-338	-313	-288
-239	-214	-189	-164	-162	-137	-112	-87	-62	-37	-12	13	38	63	88	90	115	140	165	-339	-314	-289	-264
-215	-190	-165	-163	-138	-113	-88	-63	-38	-13	12	37	62	87	112	114	139	164	-340	-315	-290	-265	-240
-191	-166	-141	-139	-114	-89	-64	-39	-14	11	36	61	86	111	113	138	163	-341	-316	-291	-266	-241	-216
-167	-142	-140	-115	-90	-65	-40	-15	10	35	60	85	110	135	137	162	-342	-317	-292	-267	-242	-217	-192
-143	-118	-116	-91	-66	-41	-16	9	34	59	84	109	134	136	161	-343	-318	-293	-268	-243	-218	-193	-168
-119	-117	-92	-67	-42	-17	8	33	58	83	108	133	158	160	-344	-319	-294	-269	-244	-219	-194	-169	-144
-95	-93	-68	-43	-18	7	32	57	82	107	132	157	159	-345	-320	-295	-270	-245	-220	-195	-170	-145	-120
-94	-69	-44	-19	6	31	56	81	106	131	156	181	-346	-321	-296	-271	-246	-221	-196	-171	-146	-121	-96

ON THE TEACHING OF ANALYTIC GEOMETRY.

By A. E. YOUNG, Miama University.

The idea of the graph, the basal concept in the development of that branch of mathematics known today as analytic geometry, was a natural outgrowth, it would seem, of the line of thought undertaken by the great philosopher and mathematician Descartes; for he undertook to parallel the algebraic operations with geomatric, and was probably the first to obtain the roots of a quadratic equation in one unknown by aid of ruler and compasses. The next step, the solution of a quadratic in two unknowns, geometrically interpreted, as in the previous case, led to the coordinate idea, and hence to the graph.

Thus was born the concept which has made possible the science of mathematics as it exists today. Algebra and geometry were thus united by Descartes and others, and since their time analysis proper and geometry have continued their development hand in hand. In fact, analytic geometry may be defined as the theory of analysis geometrically interpreted. The former is therefore a dependent science, and many mathematicians thus consider it. However, for the better understanding of analysis proper, and its applications to problems of the natural sciences, it has played an important and, oftentimes, an essential role. The sciences of astronomy, physics, and mechanics are what they are today through the theory of analysis geometrically interpreted. The engineering profession is dependent upon the ideas of analytic geometry for the applications of algebra, trigonometry, calculus, differential equations, etc., to the solution of problems. The graph is as useful to the engineer as his foot-rule, and almost as constantly used in many cases. The fundamental idea of analytic geometry, that of showing graphically the relation between two (or more) variables, the change in one and the corresponding change in the other, has crept into all professions and occupations. Many lines of work are almost entirely dependent for their recent growth upon the employment of these ideas. Students return to our technical schools at the close of every summer from factories and mills with problems of loci, etc., which uneducated mechanics have given them for solution. Many men today find themselves seriously handicapped because of their lack of knowledge of this kind and the steadily increasing demand for it. This cry of the industrial world for the solution of difficult problems can not be met in most cases without the aid of analytic geometry.

In view of the foregoing conditions, therefore, we do well to consider the question of the proper method of teaching so important a subject. What should we teach and how should we teach it, are questions which may well be asked, and the answers will vary according to the experience of different teachers.

The writer has found that analytic geometry is most difficult

for the beginning student. This difficulty is in part inherent in the subject, but to a great extent also it is due to the student's entire lack of previous training in this line of work and the failure of the teacher to appreciate this fact. This difficulty is being overcome somewhat, at present, by the introduction of graphical work in high school algebras, which is certainly a move in the right direction. It is to be hoped that in the near future coordinate geometry in its simpler forms may become a preparatory school subject. Of the year and a half now spent in the study of elementary algebra, one half year might well be given to the geometric interpretation of algebra, or, in other words, to the study of coordinates, graphs, simple loci problems, etc. This is desirable not only in order that the pupil may have the coordinate idea well grounded before he comes to college and goes into more advanced work, but also that the pupil who ends his course in the preparatory school, may have an opportunity to learn at least the elementary ideas of a subject which is of such general application in the everyday problems of life.

Given a class of students who have been grounded in the ideas of coordinate geometry, and granting that they have average intelligence, no other branch of mathematics is likely to be so interesting to both teacher and pupils. The work need never be dry to either. The problems are endless and can be graded so as to come within the reach of all. The student can see almost daily improvement in his own ability to grasp the ideas and solve the problems. A new field is open to him, whose vastness is revealed by a few suggestions from the teacher. No subject is so likely as this to turn the young, inexperienced student of mathematics into an enthusiast.

Before discussing the question of the subject matter, or the manner of presenting it, it is necessary to consider the class of students to be taught, why they are studying the subject, and what use they are to make of it. Let us divide them into two classes, the students of the technical school, and those of the college proper. The latter class may again be divided into those who elect the subject and those who take it as required work. In all colleges where mathematics is still a required subject, and there are a few outside of the technical schools, analytic geometry is usually required. Of course this is as it should be, though it may be doubted whether sufficient work of this nature is required in most colleges at the present time. A half year's work four times per week should be the lower limit, and even then the course might well be called "Coordinate Geometry," instead of "Analytic," leaving the latter name for a second course to be described later.

In a course of this kind, given to a class of students many of whom do no more work than they are compelled to do, the teacher meets with a hard problem, but one which is well worth solving, and probably no more difficult than that of teaching any other required subject in these days of electives. The difficulty of teaching required work at the present time is due largely to the ease with which a student can meet the requirements of

the elective undergraduate work in other lines. Analytical geometry is not an easy subject compared to most of the electives. In order to make the work easier for the average student, and that the best may grasp the ideas more clearly, the teacher should spend a large part of the class-period, especially near the beginning of the course, in giving explanations. Loafing on the part of students is easily prevented by requiring them to reproduce the work done by the teacher either at the close of the recitation or at the next meeting of the class, and by frequent exercises.

In regard to the material for a class taking a short course in required work, there should be little question. Cartesian coordinates should give the first introduction to the subject, but polar coordinates should be by no means omitted. The latter are quite as much employed in some lines of practical work as the former, especially in the engineering profession.

Change of coordinates and lines of reference should be discussed till the processes become familiar to the students. Loci problems of the simpler kind should be studied carefully. These problems give great difficulty to the average student, due probably, to the awkward manner in which they are presented, in many cases. If the parametric form of the equation of a curve is considered carefully in the first place, in the case of known curves, then loci problems are easily solved by getting the equation of the curve in this form and stopping there. The elimination of the parameter (or the parameters) is an algebraic question, and one which the student should be taught to consider by itself. It is the failure to distinguish between the two steps that makes the problems difficult.

It would seem that, in general, too much attention in the past has been paid, in a course like the one under discussion, to the consideration of conics and their properties. Oftentimes the dull pupils in the class, after having finally grasped the fact that there are other loci besides straight lines, at once conclude that there must of necessity be conic sections. It would be well to introduce some of the simpler curves of the third and fourth degree, in polar coordinates, as well as cartesian, and also, some of the more common non-algebraic curves such as the catenary, spirals, and cycloids, the latter in the parametric form.

No great results can be hoped for from a course of this kind given under such conditions. If the class as a whole learn thoroughly the common facts, and are at the same time aware that they have just touched the subject itself, and the really good students have the ambition to know more about it, the teacher should be well satisfied, and doubtless would be. That this latter point may be gained it would be well for the teacher throughout the course to generalize for the benefit of the better students whenever the opportunity offers.

For example, from the polar line one may go to the polar curve, giving as a particular case the polar curves corresponding to certain cubics. The ideas of class and order may be considered just enough to arouse the

curiosity of the pupils. The general method for determining asymptotes to an algebraic curve should be explained, and particular examples given. In this way we show the student some of the paths leading out, and he is naturally interested in the question as to where they lead. His ambition to know is stimulated. He begins to see how vast is the science with which he is dealing. He avoids closing the course with the idea that there were just a few points which he did not have time to learn, and he really understands that if he is to know much about the subject he must elect the course to which we now turn our attention, "elective analytic geometry," a course which should be taken up at the beginning of his Sophomore year and carried through two semesters.

We are not proposing here a course for graduate students in mathematics, but a course suited to Sophomores. In our wild attempts these days to do graduate work, we inject too much of the graduate spirit in the courses in mathematics primarily intended for the undergraduate of limited experience. The result is that the undergraduate suffers. He finds himself beyond his depth, and if he ever does touch ground before the close of the course, it is in a half drowned condition. We shall make no such mistake in outlining this course. It is taken for granted that the students who elect it have little mathematical knowledge. At most they have, or are at the same time obtaining, a little knowledge in differential calculus. To obtain the best results this should be the case.

This course should begin with a rapid review of the principal subjects considered in the previous course, namely, the straight line, the properties of conics, the transformation of coordinates, polars and tangents, etc. In connection with the study of the straight line, the idea of the point-line quality should be considered carefully and to a considerable extent. Also, line coordinates and homogeneous point coordinates should be used at the same stage of the work, and thereafter, whenever it is of advantage to do so. In connection with the idea of the tangent, the asymptote should be considered, the coordinates in use being polar as well as rectangular. The idea of the polar curve, first, second, third polar, etc., for algebraic curves should be discussed quite fully, that is, long enough to fix the ideas in the mind of the student. The cubic curves should be studied at some length. They might be divided into the five classes, cuspidal, bipartite, etc., and then the various curves in each class, characterized by the way in which they are cut by the line at infinity. No time should be spent in the careful plotting of unknown curves. Such work should be relegated to a course in calculus.

The course as outlined so far, should cover the first half of the Sophomore year and should fit the student taking it for a more advanced course in analytic geometry, which is generally called Modern Analytic Geometry. This latter course, intended for graduate students or those undergraduates who are specializing, we shall not attempt to outline here.

The great mistake, at the present time, is that of omitting a course similar to the one just outlined, and substituting in its place the one in Modern Analytic Geometry. Having taken the first, the student is infinitely better prepared for the second.

Before outlining the work of the second half year of this course, which should be in geometry of three dimensions, let us say a few words in regard to the work in analytic geometry at colleges where no mathematical work is required. Here the course should be arranged somewhat differently, we believe. The first course in analytic geometry should be made more comprehensive than when the course is required. It should include all of the work of the required course as outlined above, and some of the subjects just touched upon there should be developed more fully. A half year, four times per week, would be necessary for this work. It should be followed directly by the course in geometry of three dimensions which we are about to consider, modified somewhat to meet the new conditions.

The discussion of geometry of three dimensions should begin, of course, with a careful study of the straight line and plane, followed by work on the transformation of coordinates, change of origin and axes, etc. The discussion of the conicoids should be thorough. The general equation of the second degree should be discussed, as should also the subject of confocal conicoids. Polar and spherical coordinates should be employed in the discussion of some surfaces in which they can be used to advantage, With students who have had the required work as well as the elective course in plane analytics, the work should be done rapidly. It should be followed by an elementary discussion of space curves and algebraic surfaces. The one and two parameter idea, corresponding respectively to the curve and surface, should be touched upon sufficiently to fix the fundamental thought. The work in discussing algebraic surfaces would parallel the similar discussion of plane algebraic curves, and yield large results with little work. A class composed of students who have taken only the one course (the elective) would hardly be able to do thoroughly more than the work first outlined above.

Turning now to the teaching of analytical geometry in the Technical School, we have quite a different case for consideration. Here the classes are composed of students, the majority of whom, at least, realize that they are taking, in analytical geometry, one of the subjects which will prove most useful in their actual engineering work, and one the mastery of which is essential to success in several of the subjects which are to follow. Moreover, we can assume (though we find so many exceptions that we begin to doubt the rule) that the students have a little more mathematical knowledge when beginning the subject of analytic geometry than in the previous case.

External conditions are somewhat different from those in an academic institution. The engineering professors are demanding that the student in the technical school absorb all the mathematics that he is to have in his

first year and a half, or in the first two years at most. He thus has at best little time to assimilate his mathematical food, and we do not wonder that most of it leaves only a fleeting impression. In this connection we call attention to the books which are being written purposely to meet these conditions. We refer to books on "mathematics for engineers" in which courses like calculus, analytic geometry, etc., are combined. We have no doubt but that such a condition of affairs is bad for the student with the average preparation in mathematics.

It is two much on the face of it to expect that a freshmen, prepared as ours are at present, will be able to take up a course in "engineering mathematics," in which are combined the ideas of half a dozen subjects of which he knows little or nothing, and obtain any very definite knowledge at the end of the course. "Engineering mathematics" should *follow* thorough courses in the purely mathematical subjects and should not *replace* these. Taken thus it would prove of immense benefit to the students. Taken as it is, the principles of mathematics which are so important, and a knowledge of which is so essential to the student, are in most cases not clearly understood, if grasped at all.

We do believe, however, that a quite different course in analytic geometry should be given the prospective engineer from that provided for the ordinary student. In his case it should be always kept in mind that this is a subject which he will doubtless use in a practical way ever afterward. The difficult problem which the engineer will often meet is this: Given certain conditions, what is the equation of a curve, or curves, corresponding? Or this: Given certain corresponding values of two variables represented, say by two points in a plane, to find a relation between the variables which will approximate the true relation sufficiently near for his purpose? In the writer's short experience as a teacher in an engineering school, several graduates have reported difficulty in regard to such questions. Their great mistake seemed to be their assumption that to a set of points there corresponds necessarily one curve and only one. They had not fully grasped the facts necessary to the full understanding of loci problems, nor was it to be wondered at considering the circumstances.

As of first importance, therefore, let us urge the claims of graphical work, and the great care that should be taken by the instructor to explain most carefully the theory underlying the determination of a curve which will satisfy given conditions. Many problems of the following nature should be treated: Given a certain number of points, determine whether certain algebraic curves such as the circle, the parabola, the ellipse, and certain third degree curves, pass through them. Take up cases where one arbitrary constant remains in the equation, impose another condition, etc,, etc. Consider the determination of loci whose equations hall be of some of the more common non-algebraic forms, as the catenary, spirals, or the trochoids. Consider the form of the equations of some of these if the origin and axes

are changed. Be careful to see that the pupils have a great familiarity with the ideas involved in the transformation of coordinates.

The next most important part of their work should be the plotting of curves from given equations. A student who left college at the close of his junior year, reported after being out three years, engaged in construction work for one of our large railroad companies, that the study of "higher plane curves," had proved of the greatest use to him of any of his mathematical work. The ideas gained enable him to treat his problems intelligently. This sort of work, however, cannot be done without using the ideas and methods of the calculus. Partly for that reason, and partly in order that the student may begin early the study of this great subject, we would urge that no extended course be taken in analytical geometry, apart from calculus. A semester's work should enable the student to grasp the fundamental ideas of the subject of plane and solid analytic geometry. The course should then continue, but as a course in differential calculus, geometrically applied. It is a mistake to fill a book on elementary calculus with problems from all the sciences in which it may be applied. Much better would it be to confine the applications almost entirely to the science of elementary analytical geometry. In this way the student will stand a much better chance of actually grasping the principles of the calculus and will at the same time learn a great deal, and in the proper way, about a very necessary subject. It may be that when the student takes up mechanics and allied subjects, he will not start off so briskly at first but in the end he should do much better work, as he should have the principles of calculus in better shape and hence be able to apply them better. An engineer of long experience, in stating that he never made any use of calculus in his work, seemed to think that it was due to the way in which the subject had been presented to him. No doubt he was right about it, although we might disagree as to the way it should be presented. An intelligent, well-educated engineer *should* use calculus to his advantage in some of his work. In fact, the gentleman referred to above did say that engineers in the United States Government service do use it continually. A man can not use calculus understandingly unless he is familiar with the *theory* of calculus. A careful study of the theory should precede its general application, and to the fact that this is not the case in many of our technical schools, we owe the "failure of the science" as an engineering tool. It would seem that today the mathematicians are falling into this very error themselves, urged on by the professors in the applied sciences. They say to themselves, "I must not teach any pure mathematics," and they almost apologize to their classes for wasting any time on calculus itself, or analytic geometry before applying it. The result is a great deal like that of beginning the study of Greek literature before learning the Greek language. They make a mistake in this, and some of our leading educators in large engineering schools are becoming convinced of this fact. They say, "Stick to your own science. Give the

students the pure mathematical ideas, and let the professors of physics, mechanics, etc., do the applying of the principles." Much better would it be to give the non-engineering student work in the broader applications of the mathematical theory, for the engineer is sure to get this work, and the other students will have no other chance.

Finally, in a course in analytic geometry for engineers, we would urge again the general use of the parametric form of the equation. It has a wide application and is of common use in their profession, as, for example, in the case of the moving point whose coordidates are functions of the time. Polar coordinates, also, are almost as familiar to the engineer in much of his work as rectangular, and should receive careful attention in the course in analytic geometry, both before and after the application of the theory of calculus.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

323. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that the relation $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$ can hold for real numbers only when they are in proportion.

Solution by the PROPOSER.

Expanding and rearranging, we get

$$a^2(c^2+d^2)-2abc(b+d)+b^4+b^2c^2-2bc^2d+b^2d^2+c^4=0.$$

Solving for a,

$$a=\frac{bc(b+d)\pm(c^2-bd)\sqrt{[-(b^2+c^2+d^2)]}}{c^2+d^2}$$

which can be real only when $c^2-bd=0$, *i. e.* when $c/d=b/c$. Then

$$a=\frac{bc\ (b+d)}{c^2+d^2}=\frac{bc\ (b+d)}{bd+d^2}=\frac{bc}{d},$$

since $b+d\neq 0$. Therefore, $a=\dfrac{bc}{d}$ or $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{b}{c}$.

Also solved by G. B. M. Zerr.

324. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the *finite* series

$$\frac{16n^2-2^2}{4!}-\frac{(16n^2-2^2)(16n^2-4^2)}{6!}+\frac{(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}-\ldots$$

where n is a positive integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

$$\cos m\theta=1-\frac{m^2}{2!}\sin^2\theta+\frac{m^2(m^2-2^2)}{4!}\sin^4\theta-\frac{m^2(m^2-2^2)(m^2-4^2)}{6!}\sin^6\theta+\ldots$$

Let $m=4n$, then

$$\cos 4n\theta=1-\frac{16n^2}{2!}\sin^2\theta+\frac{16n^2(16n^2-2^2)}{4!}\sin^4\theta$$

$$-\frac{16n^2(16n^2-2^2)(16n^2-4^2)}{6!}\sin^6\theta+\ldots$$

Let $\theta=\frac{1}{2}\pi$, then $\cos 4n\theta=\cos 2n\pi=1$.

$$\therefore 1=1-\frac{16n^2}{2!}+\frac{16n^2(16n^2-2^2)}{4!}-\frac{16n^2(16n^2-2^2)(16n^2-4^2)}{6!}$$

$$+\frac{16n^2(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}-\ldots$$

$$\therefore \tfrac{1}{2}=\frac{(16n^2-2^2)}{4!}-\frac{(16n^2-2^2)(16n^2-4^2)}{6!}$$

$$+\frac{(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}+\ldots$$

GEOMETRY.

345. Proposed by LLOYD HOLSINGER, Bradley Polytechnic Institute, Peoria, Ill.

If a variable polygon move in such a way that its n sides turn severally round n fixed points O_1, O_2, ..., O_n while $n-1$ of its vertices slide, respectively, along $n-1$ fixed straight lines v_1, v_2, ..., v_{n-1}, then the last vertex will describe a conic; and the locus of the point of intersection of any pair of non-adjacent sides will also be a conic. Cremona's *Projective Geometry.*

Solution by HOWARD C. FEEMSTER, A. B., York College, York, Neb.

Let the sides of the polygon a_1, a_2, ..., a_n turn severally around the n fixed points O_1, O_2, ..., O_n, describing n flat pencils, O_1, O_2, ..., O_n, while the $n-1$ vertices A_1, A_2, ..., A_{n-1} slide, respectively, along the $n-1$ fixed straight lines, v_1, v_2, ..., v_{n-1}, forming $n-1$ ranges, A_1, A_2, ..., A_{n-1}. Pencils O_1 and O_2 are perspective with range A_1; ranges A_1 and A_2 are perspective with pencil O_2; pencils O_2 and O_3 are perspective with range A_2, thus making the n pencils and $n-1$ ranges projective.

Now the projective pencils O_1 and O_n or any two of the non-consecutive pencils (using above order) will not, in general, be perspective, and every such intersection will thus describe a conic range as its locus. *Encyclopedia Britannica,* "Projective Geometry," Sec. 45.

347. Proposed by W. J. GREENSTREET, M. A. Marling School, Stroud, England.

ABC is a triangle, and *D, E, F,* are the mid points of the arcs of its nine-point circle cut off by *BC, CA, AB,* respectively. The inscribed circle touches these sides at *X, Y, Z.* Are the lines *DX, EY, FZ* concurrent? A purely geomatrical discussion required.

Solution by BENJ. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

In my *Mathematical Solution Book,* page 461, it is shown that the perpendiculars to the chords cut off by the nine-point circle, at the mid-points intersect in a point *S,* which is the center of the nine-point circle. These perpendiculars produced from the chords to the subtended arcs, bisect the smaller arcs in *D, E,* and *F.* Since the in-circle is tangent to the chords or chords produced in *X, Y, Z,* perpendiculars erected at these points meet in a point, *I,* the center of the in-circle. Drawing the lines *DX* and *EY,* and producing them until they meet *IS,* or *IS* produced, in *C* and *C'*, respectively, we have the triangles *SDC* and *SDC'*.

The triangles *SDC* and *IXC* are similar, as are also *SDC'* and *IYC'*. Hence, *SD,* (the radius of the nine-point circle): *IX,* (the radius of the in-circle) $=SC : IC$; and similarly, *SE,* the radius of the nine-point circle: *IY,* the radius of the in-circle, $=SC' : IC'$. Hence, $SD-IX : IX=SC-IC(=SI) : IC$, and $SE-IY : IY=SC'-IC'(=SI) : IC'$. Hence, $IC=IC'$ and, therefore, *DX* and *EY* intersect the line joining the nine-point circle and the in-circle in the common point *C.* In the same way, it may be shown that the line *FZ* intersects the same line in the point *C.* Hence, the three lines are concurrent.

CALCULUS.

275. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Explain fully why the circular measure of an angle is used in the calculus.

Solution by the PROPOSER.

There is one good reason for changing from the *gradual measure* to the *circular measure* in the calculus.

In finding the derivative of sin$u$ we are required to evaluate

$$\lim_{u \doteq 0} \left| \frac{\sin\frac{1}{2}u}{\frac{1}{2}u} \right|,$$

which becomes equal to unity.

Had we used degrees, we would have introduced the extraneous factor $\pi/360$. For, suppose u to be given in degrees, minutes or seconds; then we have

$$\frac{d}{dx}(\sin u^\circ)=\frac{\pi}{180}\cos u^\circ \frac{du}{dx};\quad \frac{d}{dx}(\sin x')=\frac{\pi}{180.60}\cos x';$$

$$\frac{d}{dx}(\sin x'')=\frac{\pi}{180.60.60}\cos x''.$$

Taking a numerical example, let $x=20^\circ$ and $\triangle x=.0001^\circ$.

From a table of natural sines, radius being 1, we obtain $\sin 20^\circ=.3420201=u$, say; and, $\sin 20.0001^\circ=.34202174=u+\triangle u$. $\therefore \triangle u=.00000164$.

Now, the length of a circular arc subtending an angle of $.0001^\circ$ is, (radius being 1) $.0000017453=\triangle x$.

$$\therefore \frac{\triangle u}{\triangle x}=\frac{\text{increment of sine}}{\text{increment of arc}}=\frac{.00000164}{.0000017453}=.93967,$$

which is the value of $\cos 20^\circ$ correct to four places. The actual value, found by indefinitely decreasing the increment, is

$$\frac{du}{dx}=\cos 20^\circ=.9396926.$$

Remark by BENJAMIN F. FINKEL.

An angle, being a measurable magnitude, must be measured, or compared, by some arbitrarily chosen magnitude of its own kind. This arbitrarily chosen angular magnitude, we call the *unit angle.* In geometry, the unit angle is the angle formed by two straight lines perpendicular to each other. This unit is too large for trigonometric purposes, and so a smaller angle is chosen as a unit, namely, the 90th part of a right angle, called a degree. Any other arbitrarily chosen part would serve analysis as well, though the convenience or inconvenience in computation would be modified

thereby. Now in elementary geometry it is proved that angles at the center are proportional to the intercepted arcs.

Let $s[L]$, $r[L]$, and θ be the arc, radius, and center angle, s and r being measured in the same *length-unit*, the angle unit being at present undetermined. Then $s[L]=kr[L]\theta$. Let us now *assume* $\theta=1$, *the unit angle, when* $s=r$. Hence, $k=1$, under this assumption. We thus have as our unit angle, $[\Theta]$, the angle at the center of a circle, which intercepts an arc equal in length to the radius of the circle. Hence, we have the general relation $s[L]=r[L]\,\theta\,[\Theta]$ and $\theta[\Theta]=s/r$. This unit angle, $[\Theta]$, is called a *radian*. In terms of the fundamental units of length, mass, and time, it is of zero dimensions, since $[\Theta]=\frac{s[L]}{r[L]}[M^\circ][T^\circ]=L^\circ M^\circ T^\circ$, s being equal to r. From this we see that the great advantage of this unit of angular measure over that of any other is that it avoids, as remarked by Mr. Schmall, the introduction of a *coefficient of variation* different from unity.

PROBLEMS FOR SOLUTION.

ALGEBRA.

329. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Between the quantities a and b there are inserted n arithmetical and n harmonical means, and a series of n terms is formed by dividing each arithmetical by the corresponding harmonical mean. Show that the sum of the series is, $n\left[1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right]$.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n\doteq\infty} x^{1/(2n-1)},\ n=1,2,\ldots;\quad f(x)=\lim_{n\doteq\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

GEOMETRY.

357. Proposed by E. R. HOYT, St. Louis, Mo.

A room is 30 feet long, 12 feet wide, and 12 feet high. At one end of the room, 3 feet from the floor, and midway from the sides, is a spider. At the other end, 9 feet from the floor, and midway from the sides, is a fly. Determine the shortest path by way of the floor, ends, sides, and ceiling, the spider can take to capture the fly.

358. **Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Neb.**

Cut four coplanar non-copunctual straight lines in a harmonic range.

CALCULUS.

387. **Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.**

An object P, being placed beyond the principal focus F of a convex lense, determine its position when its distance PQ, from its image Q, is a minimum.

388. **Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.**

Find $\int \frac{xdx}{(1+x^3)^{2/3}}$.

MECHANICS.

240. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

A simple beam length $2a$, supported at both ends, is loaded in the form of a parabola, height of vertex b. Find deflection at center due to this load.

241. **Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.**

In a certain New York theatre there is an asbestos curtain supported by thin circular rings, radius r, which move on a cylindrical rod of radius a. The curtain is intended to be drawn by a *steady pull*. Taking μ as the coefficient of friction, show that this will not be possible if r be less than $a\sqrt{(1+\mu^2)}$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

170. **Proposed by PATRICK WALSH, 1451 Annunciation Street, New Orleans, La.**

The areas of rectangles A and B are respectively 15170 10/27 and 31230.3627. Find the sides and diagonal of each rectangle in exact or rational numbers.

NOTES AND NEWS.

The editors extend to all our readers the Greetings of the Coming Year. We hope that delays in the regular appearance of THE MONTHLT will not occur during 1910. We trust that our subscribers will continue their support and that we may have the immediate renewals of those who have not already remitted their subscriptions for the new year.

On December 23, Editor Miller was married to Miss Cassie A. Boggs. They will make their home in Urbana, Illinois. The other editors and all readers of the monthly extend congratulations to the happy couple.